Limit Dynamics

Apart from phrases like for *every* number $\epsilon > 0$ (i.e. $\forall\epsilon > 0$), for *every* number $M > 0$ (i.e. $\forall M > 0$), there is nothing in the classical definition of a limit that appeals to the notion of a completed infinity. However, the completed infinity of the real numbers is a prerequisite with the common definition of a limit. Therefore we have another proposal. Suppose that the underlying substrate of real numbers can be regarded instead as a *type*, like in modern programming languages, meaning that you can create any real number at will, as soon as you need it, without having all real numbers being present, as a completed infinity, in a finished set. Then the limit concept as such is completely finitistic. And $(\forall\epsilon \in \mathbb{R}+)$ could be regarded as a *figure of speech.*

The Limit Concept

Disclaimer. The following can be found in numerous standard texts on Calculus, such as: James Stewart 5e, Calculus early transcendentals, Thomson, ISBN 0-534-27409-9.

`http://en.wikipedia.org/wiki/Limit_(mathematics)`

Suppose f is a real-valued function of x and a is a real number. The expression:

$$\lim_{x \to a} f(x) = L$$

means that $f(x)$ can be made as close to L as desired, by making x sufficiently close to a, but without actually letting x be a. In this case, we say that "the limit of $f(x)$, as x approaches a, is L".
The exact definition is as follows. Let $f(x)$ be a function defined on an interval that contains $x = a$. Then we say:

$$\lim_{x \to a} f(x) = L$$

if for every number $\epsilon > 0$ there is some number $\delta > 0$ such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

Note that $f(x)$ is possibly not defined at $x = a$.
As an example, consider the following limit:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Indeed. Because $a = 1$, but $0 < |x - a|$, so $x \neq 1$, we may safely write:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \to 1}(x + 1)$$

Now take $\delta = \epsilon$, then there always exists some number δ, namely ϵ, such that $|(x + 1) - 2| = |x - 1| < \epsilon$ whenever $|x - 1| < \delta$.
Closely related to the definition of a limit, but not identical to it, is the definition of continuity. A function f is *continuous* at a number a if:

$$\lim_{x \to a} f(x) = f(a)$$

Example. Let $f(x)$ be defined by:

$$f(x) = \begin{cases} 2 & \text{for} \quad x = 1 \\ (x^2 - 1)/(x - 1) & \text{for} \quad x \neq 1 \end{cases}$$

Then we have proved that $f(x)$ is continuous at $x = 1$. If we define instead:

$$f(x) = \begin{cases} 0 & \text{for} \quad x = 1 \\ (x^2 - 1)/(x - 1) & \text{for} \quad x \neq 1 \end{cases}$$

Then $f(x)$ is *not* continuous at $x = 1$, because: $\lim_{x \to 1} f(x) = 2 \neq 0$.
Suppose, again, that f is a real-valued function of x. The expression:

$$\lim_{x \to \infty} f(x) = L$$

means that $f(x)$ can be made as close to L as desired, by making x large enough. Without actually letting x be ∞ would be a trivial addendum in this case. We say that "the limit of $f(x)$, as x approaches infinity, is L". The exact definition is as follows. Let $f(x)$ be a function defined for sufficiently large values of x. Then we say:

$$\lim_{x \to \infty} f(x) = L$$

if for every number $\epsilon > 0$ there is some number $N > 0$ such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x > N$$

As an example, consider the following limit:

$$\lim_{x \to \infty} 1/x = 0$$

We find that this limit is equal to zero. Indeed:

$$|1/x - 0| < \epsilon \quad \text{whenever} \quad x > N \quad \text{with} \quad N = 1/\epsilon$$

Last but not least we have *infinite limits*. Symbolically:

$$lim_{x \to \infty} f(x) = \infty$$

if for every number $M > 0$ there is some number $N > 0$ such that:

$$f(x) > M \quad \text{whenever} \quad x > N$$

But, in some circles, such infinite limits are simply said *not to exist*.
Claimer. The following will *not* be found in any standard texbook on calculus. Let's restrict attention to the first limit definition given above:

$$\lim_{x \to a} f(x) = L$$

if for every number $\epsilon > 0$ there is some number $\delta > 0$ such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

Numbers like $\epsilon > 0$ and $\delta > 0$ are known in computational work as *errors*. Even the real numbers themselves in a digital computer are not error free; as is obvious if one seeks to represent irrational numbers such as π or $\sqrt{2}$. If x is the floating point number representing π and δ is the "machine eps" (i.e. an error) then only the following is true - it reminds of something that we have seen in the section *Reals in Reality*:

$$x \not\equiv \pi \quad \text{and} \quad 0 < |x - \pi| < \delta$$

Even worse. While carrying out computations with real machine numbers, errors tend to accumulate. This is actually reflected within the (δ, ϵ) formalism for limits. Let, for example, $|x - a| < \delta_x$ and $|y - a| < \delta_y$, where (a, b) are supposed to be "exact" and (x, y) are machine numbers. Next calculate:

$$|(x + y) - (a + b)| = |(x - a) + (y - b)| \le |x - a| + |y - b| < \delta_x + \delta_y$$

So the error in the sum $(x + y)$ is most probably greater than one of the errors (δ_x, δ_y) , namely the sum of these. Similar expressions can easily be derived for the other elementary operations. And are encountered as well in common proofs involving limits. We may conclude that the concept of a limit actually represents *error processing* - though in an *idealized* manner. To put it the other way around: the *materialization* of the limit concept is error processing.
Now take another look at the limit definition, especially the clause "if for every number $\epsilon > 0$ there is some number $\delta > 0$ such that". If $f(x)$ is interpreted as (the end result of) a calculation, then it simply says that the error propagation of that calculation, though dependent upon initial errors δ, must be guaranteed to be limited within a certain pre-defined bound ϵ.
Compare all this with the Approximate Equality definition for real numbers, as has been developed in the section *Infinitesimal Equality*:

$$(x \approx y) \;:\Longleftrightarrow\; \left[\left(x \stackrel{T}{=} y \right) \vee \left(0 \stackrel{T}{<} |x - y| \stackrel{T}{<} \delta \right) \right] \qquad \text{where} \quad \delta \stackrel{S}{=} 0$$

Upon careful inspection of the Limit definition the following pattern is observed:

- $[\ f(x) \approx L\ ]$ but not $\left[\ f(x) \stackrel{T}{=} L\ \right]$ whenever $[\ x \approx a\ ]$ but not $\left[\ x \stackrel{T}{=} a\ \right]$

With other words: the (δ, ϵ) inequalities in the limit definition are an idealization of (approximate) equalities, but *without an identity*. Because their identity is lacking, numbers are equal to themselves but, strange as it seems, they are not identical to themselves. The *materialization* of a limit gives the second part of the approximate equality definition; it is a *Infinitesimal Equality*. Putting it the other way around: limits are (the *idealization* of) infinitesimal equalities. That's why limits are very much qualified for the description of *change*, as is exemplified with the use of derivatives dependent on time, in kinematics and mechanics.

Iterated Limits

Little trouble is encountered with limits, as long as they are *single*, that is: not superposed upon each other. The situation becomes radically different, however, with repeated or *iterated limits*.

Example 1

We start with an example that could easily come from a standard textbook on calculus.

$$F(x,y) = \begin{cases} (x^2 - y^2)/(x^2 + y^2) & \text{for} \quad (x,y) \neq (0,0) \\ 0 & \text{for} \quad (x,y) = (0,0) \end{cases}$$

Let's proceed with some - innocent looking - iterated limits.

$$\lim_{x\to\infty} \left(\lim_{y\to\infty} \frac{x^2 - y^2}{x^2 + y^2} \right) = \lim_{x\to\infty} \left(\lim_{y\to\infty} \frac{x^2/y^2 - 1}{x^2/y^2 + 1} \right) = \lim_{x\to\infty} (-1) = -1$$

$$\lim_{y\to\infty} \left(\lim_{x\to\infty} \frac{x^2 - y^2}{x^2 + y^2} \right) = \lim_{y\to\infty} \left(\lim_{x\to\infty} \frac{1 + y^2/x^2}{1 + y^2/x^2} \right) = \lim_{x\to\infty} (+1) = +1$$

The iterated limits, apparently, *do not commute*.

$$\lim_{x\to 0} \left(\lim_{y\to 0} \frac{x^2 - y^2}{x^2 + y^2} \right) = \lim_{x\to 0} \left(\frac{x^2}{x^2} \right) = +1$$

$$\lim_{y\to 0} \left(\lim_{x\to 0} \frac{x^2 - y^2}{x^2 + y^2} \right) = \lim_{y\to 0} \left(\frac{-y^2}{y^2} \right) = -1$$

The iterated limits, apparently, *do not commute*.
Now consider again the iterated limits, from a *top down* viewpoint, leading to a level of mathematics somewhat below the idealization, i.e. material mathematics:

$$\lim_{x\to 0} \left(\lim_{y\to 0} \frac{x^2 - y^2}{x^2 + y^2} \right) = \lim_{x\to 0} \left(\frac{x^2}{x^2} \right)$$

But wait! The **Axiom** comes into effect here: *Completed infinity is not given.* The limit for $y \to 0$ therefore may not be considered as completed. Or the value $y = 0$, upon retrospect, is not actually reached: $y \neq 0$. Therefore what's actually left, instead of the number 0, is a number slightly different from 0, let's call it ϵ:

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2-y^2}{x^2+y^2}\right) = \lim_{x\to 0}\left(\frac{x^2-\epsilon^2}{x^2+\epsilon^2}\right) = \left(\frac{-\epsilon^2}{+\epsilon^2}\right) = -1$$

This is the same outcome as with:

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2-y^2}{x^2+y^2}\right) = \lim_{y\to 0}\left(\frac{-y^2}{y^2}\right) = -1$$

But wait! The **Axiom** comes into effect here: *Completed infinity is not given.* The limit for $x \to 0$ therefore may not be considered as completed. Or the value $x = 0$, upon retrospect, is not actually reached: $x \neq 0$. Therefore what's actually left, instead of the number 0, is a number slightly different from 0, let's call it δ:

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2-y^2}{x^2+y^2}\right) = \lim_{y\to 0}\left(\frac{\delta^2-y^2}{\delta^2+y^2}\right) = \left(\frac{\delta^2}{\delta^2}\right) = +1$$

This is the same outcome as with:

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2-y^2}{x^2+y^2}\right) = \lim_{x\to 0}\left(\frac{x^2}{x^2}\right) = +1$$

We conclude that:

$$\lim_{x\to 0}\left(\lim_{y\to 0}F(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0}F(x,y)\right)$$

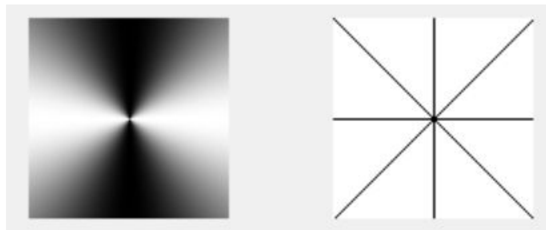
So it seems that the iterated limits do indeed *commute* in the first place. But the outcome can be $+1$ or -1, seemingly at will. The latter would lead to the conclusion that both iterated limits, actually, *do not exist.*
It's good to take a closer look at the above. We will do so by introducing polar coordinates:

$$x = r\cos(\phi) \quad ; \quad y = r\sin(\phi)$$

Giving:

$$F(x,y) = \frac{x^2-y^2}{x^2+y^2} = \frac{r^2(\cos^2(\phi)-\sin^2(\phi))}{r^2(\cos^2(\phi)+\sin^2(\phi))} = \cos(2\phi)$$

Now we can understand immediately why those limits of $F(x,y)$ for x and y *indeed* do not commute. At $(x,y) = (0,0)$ this function is *singular* and it assumes *any* value between -1 and $+1$ there. But the latter is the case anywhere else in the $(x,y)-$plane. A limit for $x \to \infty$ and/or $y \to \infty$ can assume *any* value between -1 and $+1$ as well. The function is sort of a circular wave and has a period π, with the radii of a circle as its contour lines. A few pictures say more than a thousand words.



Black corresponds with function values (-1). White corresponds with function values $(+1)$. Grey corresponds with function values between these extremes. Note that the contour lines intersect at the origin, something that can only happen if the function is singular at that place.
A "natural" way to define limits with the function $F(x,y)$ is to proceed along contour lines. Just suppose that we decide to approach the origin as well as infinity along a single contour line. What we have then are single limits.
Select a contour line in the "black" region of $F(x,y)$, namely $x = 0$. Then:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

$$\lim_{(x,y)\to(\infty,\infty)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y\to \infty} \frac{-y^2}{y^2} = -1$$

Select a contour line in the "white" region of $F(x,y)$, namely $y = 0$. Then:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = +1$$

$$\lim_{(x,y)\to(\infty,\infty)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to \infty} \frac{x^2}{x^2} = +1$$

Select a contour line in the "grey" region of $F(x,y)$, namely $y = \pm x$. Then:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2 - x^2}{x^2 + x^2} = 0$$

$$\lim_{(x,y)\to(\infty,\infty)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to \infty} \frac{x^2 - x^2}{x^2 + x^2} = 0$$

Select another contour line in the "grey" region of $F(x,y)$, namely $y = x/\sqrt{2}$. Then:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2 - x^2/2}{x^2 + x^2/2} = \frac{1}{3}$$

$$\lim_{(x,y)\to(\infty,\infty)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to \infty} \frac{x^2 - x^2/2}{x^2 + x^2/2} = \frac{1}{3}$$

Example 2

So far so good. We first continue with another standard textbook example.

$$G(x,y) = \begin{cases} 2xy/(x^2+y^2) & \text{for} \quad (x,y) \neq (0,0) \\ 0 & \text{for} \quad (x,y) = (0,0) \end{cases}$$

Proceeding again with some - innocent looking - iterated limits:

$$\lim_{x\to\infty}\left(\lim_{y\to\infty}\frac{2xy}{x^2+y^2}\right) = \lim_{\to\infty}\left(\lim_{y\to\infty}\frac{2x/y}{x^2/y^2+1}\right) = \lim_{x\to\infty}(0) = 0$$

$$\lim_{y\to\infty}\left(\lim_{x\to\infty}\frac{2xy}{x^2+y^2}\right) = \lim_{y\to\infty}\left(\lim_{x\to\infty}\frac{2y/x}{1+y^2/x^2}\right) = \lim_{x\to\infty}(0) = 0$$

The iterated limits, apparently, commute.

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{2xy}{x^2+y^2}\right) = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{2y/x}{1+y^2/x^2}\right) = \lim_{x\to 0}\left(\frac{0}{1}\right) = 0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{2xy}{x^2+y^2}\right) = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{2x/y}{x^2/y^2+1}\right) = \lim_{y\to 0}\left(\frac{0}{1}\right) = 0$$

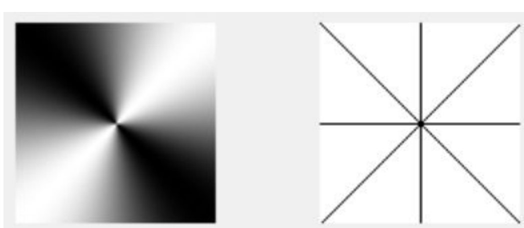
The iterated limits, apparently, commute. But it's good to take a closer look at the above. We will do so by introducing polar coordinates:

$$x = r\cos(\phi) \quad ; \quad y = r\sin(\phi)$$

Giving:

$$G(x,y) = \frac{2xy}{x^2+y^2} = \frac{r^2.2\cos(\phi)\sin(\phi)}{r^2(\cos^2(\phi)+\sin^2(\phi))} = \sin(2\phi)$$

Unlike what one might expect, the function $G(x,y)$ doesn't behave much neater than than function $F(x,y)$ in the previous example. The iterated limits for x and y do indeed commute, but such is more or less a coincidence. In fact, the function $G(x,y)$ is like the function $F(x,y)$ but rotated over an angle of 45^o (counterclockwise), because $\sin 2\phi = \cos 2(\phi - \pi/4)$:



A "natural" way to define limits with the function $G(x,y)$ is to proceed along contour lines. Just suppose that we decide to approach the origin as well as infinity along a single contour line. What we have then are single limits.

So let's repeat the limit for $G(x,y)$ while assuming that $y=-x$. The latter is a "black" contour line extending to infinity and to zero. Then we get:

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{-2x^2}{2x^2} = -1$$

$$\lim_{(x,y)\to(\infty,\infty)} \frac{2xy}{x^2+y^2} = \lim_{x\to \infty} \frac{-2x^2}{2x^2} = -1$$

Repeat the limit for $G(x,y)$ while assuming that $y=+x$, a "white" contour line extending to infinity and to zero. Then we get:

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2x^2}{2x^2} = +1$$

$$\lim_{(x,y)\to(\infty,\infty)} \frac{2xy}{x^2+y^2} = \lim_{x\to \infty} \frac{2x^2}{2x^2} = +1$$

These outcomes are to be expected, because the function $G(x,y)$ is like the function $F(x,y)$ as described above, apart from a rotation of the coordinate system. Indeed, we have seen the following.
Select a contour line in the "black" region of $F(x,y)$, namely $x=0$. Then:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

$$\lim_{(x,y)\to(\infty,\infty)} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to \infty} \frac{-y^2}{y^2} = -1$$

Select a contour line in the "white" region of $F(x,y)$, namely $y=0$. Then:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = +1$$

$$\lim_{(x,y)\to(\infty,\infty)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to \infty} \frac{x^2}{x^2} = +1$$

Therefore attention can be restricted to $F(x,y)$, without missing anything. With $F(x,y)$, the function values vary between -1 and $+1$, meaning that limits like the above, *without* further specification (e.g. along a contour line), actually do not exist for $G(x,y)$ as well:

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \text{undefined}$$

$$\lim_{(x,y)\to(\infty,\infty)} \frac{2xy}{x^2+y^2} = \text{undefined}$$

Example 3

Let n be positive integer and x be real. Consider the following function:

$$f_n(x) \begin{cases} = 1 & \text{if} \quad n \le x < n+1 \\ = 0 & \text{otherwise} \end{cases}$$

Its easy to see that:

$$\int_{-\infty}^{+\infty} f_n(x)\, dx = 1$$

And hence:

$$\lim_{n\to\infty} \left[\int_{-\infty}^{+\infty} f_n(x)\, dx \right] = 1$$

as well, while

$$\lim_{n\to\infty} f_n(x) = 0$$

for every x, and so

$$\int_{-\infty}^{+\infty} \left[\lim_{n\to\infty} f_n(x) \right] dx = 0$$

Conclusion: the integral and the limit, apparently, *do not commute.*
Let's analyze this further, nevertheless. We shall first unravel the meaning of the infinities involved:

$$\int_{-\infty}^{+\infty} f_n(x)\, dx = \lim_{L\to\infty} \int_{-L}^{+L} f_n(x)\, dx$$

Therefore what's really going on is the following (n natural, L real):

$$\lim_{n\to\infty} \lim_{L\to\infty} \int_{-L}^{+L} f_n(x)\, dx \left(= \lim_{n\to\infty} 1 = 1 \right)$$

$$\lim_{L\to\infty} \lim_{n\to\infty} \int_{-L}^{+L} f_n(x)\, dx \left(= \lim_{n\to\infty} 0 = 0 \right)$$

Where the answer between parentheses is the one that is most probably expected by a maths teacher. With other words: what we have is an *iterated limit.* Here comes an ASCII picture that is associated with this little problem:

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                              ---------
 -----------------------------|       |_______________________ f_n
                  L           n  L    n+1      L
```

Apart from the limits. As long as we are in the finitary domain, then it's obvious that there exist the following possibilities for L and n:

$$L < n \quad ; \quad L = n \quad ; \quad n < L < n+1 \quad ; \quad n+1 = L \quad ; \quad n+1 < L$$

If we proceed by taking the limit for $L \to \infty$ first, then it seems as if $L > n+1$ is the final fact. This is often expressed as "keep n fixed and let L become

infinite". If we act in this way, then the outcome is indeed 1. Note, however, that it is assumed silently here that the limit with L is in some sense *completed*. Therefore, no matter how we proceed next by taking the limit for $n \to \infty$, there is no way anymore to accomplish $n+1$ to become greater than L eventually.
If we proceed by taking the limit for $n \to \infty$ first, then it seems as if $L < n$ is the final fact. This is often expressed as "keep L fixed and let n become infinite". If we act in this way, then the outcome is indeed 0. Note, however, that it is assumed silently here that the limit with n is in some sense *completed*. Therefore, no matter how we proceed next by taking the limit for $L \to \infty$, there is no way anymore to accomplish L to become greater than n eventually.
As far as the abovementioned utterings with "fixed" are concerned, I would rather say that a "fixed" real L (or natural n) is an undefined concept. A real is just a real and there's nothing "fixed" or "variable" with it. But it could be that such is only a figure of speech and should not be taken too literally.
A more serious objection is provided by adhering to the **Axiom** that *Completed infinity is not given*. Because then, if we let L go to infinity, then the end result is a finite, though possibly very large, number. Meaning that it is still possible for n to become even larger. And if we let n go to infinity, then the end result again is a finite, though possibly very large, number. Meaning that it is still possible for L to become even larger. In short, if the Axiom is accepted, then we have to live with the fact that *both* n and L approach limiting values in whatever fashion. What is the proper way to express this idea? The following is our proposal:

$$\lim_{\min(n,L)\to\infty} \int_{-L}^{+L} f_n(x)\, dx$$

And now the answer is that this limit: *does not exist*. Or rather that it has any value between 0 and 1, not by coincidence the two extremes found by students who got an 'A' for their exam.
That's not the last word about it, though. Let's repeat. It's obvious that there exist the following possibilities for L and n:

$$L \le n \quad ; \quad n < L < n+1 \quad ; \quad n+1 \le L$$

Now if we *keep one of these conditions* while taking the limit for $(n, L) \to (\infty, \infty)$, then the iterated limit *does* exist for two of the following three cases:

$$\lim_{(n,L)\to\infty} \int_{-L}^{+L} f_n(x)\, dx = \begin{cases} 0 & \text{for} \quad L \le n \\ \text{undefined} & \text{for} \quad n < L < n+1 \\ 1 & \text{for} \quad n+1 \le L \end{cases}$$

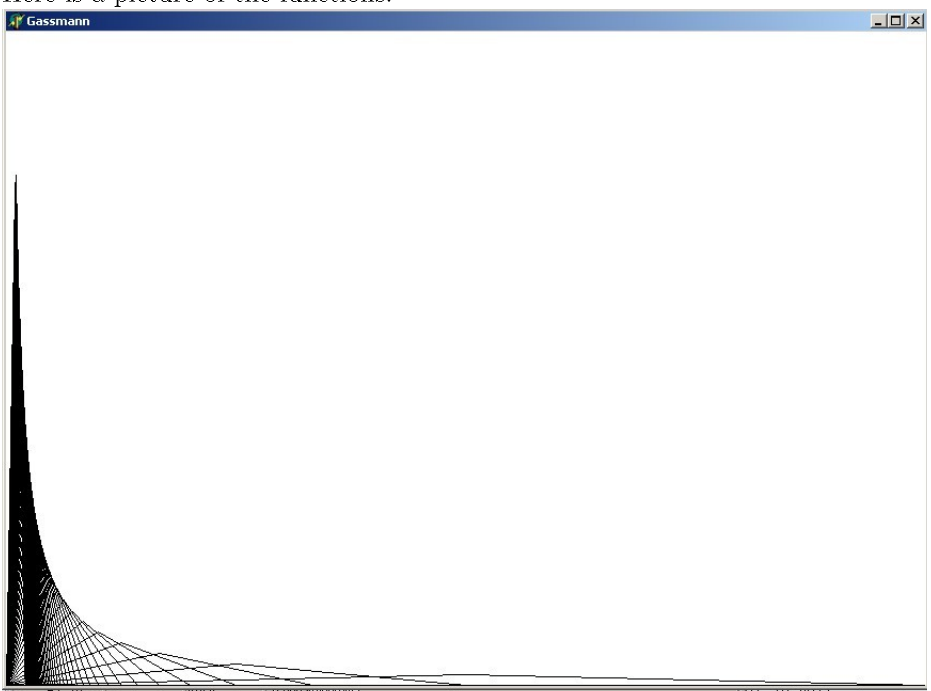
Where *undefined*, in this case, means that the limit has any value between 0 and 1.

Example 4

Consider the functions $g_n(x)$, with $n \in \mathbb{N}$, $n \geq 1$ and $x \in \mathbb{R}$, defined as follows:

$$g_n(x) = \begin{cases} 2.n^2.x & \text{if} \quad 0 \leq x < 1/(2n) \\ 2n - 2.n^2.x & \text{if} \quad 1/(2n) \leq x < 1/n \\ 0 & \quad\;\; \text{everywhere else.} \end{cases}$$

Standard argument:
These functions are triangular, and they all disappear outside of $[0, 1]$*, so I can compute* $\int_0^1 g_n(x)\,dx = 1$ *for every* n*. For every* x*,* $\lim_{n\to\infty} g_n(x) = 0$*. So again, the limit and the integration can't be interchanged.*
Here is a picture of the functions:



We see that the function $g_n(x)$ becomes a sharp peak at $x = 0$ for $n \to \infty$ and that, geometrically, it certainly does not disappear or becomes zero. The devil is again in the false doctrine that it would be possible to "fix" x and let n go to infinity. Again, we say that such a "fixed" real x is an *undefined* concept. A real is just a real and there's nothing "fixed" or "variable" with it. Such is even more obvious with the present function $g_n(x)$, where x and n are clearly intermixed, by definition. So we *can* have: $0 \leq x < 1/(2n), 1/(2n) \leq x < 1/n, 1/n \leq x$ and nothing prevents us from going to limits *while* this is the case. If we just do this, then what we get is what physicists know as a delta function. Informally:

$$\delta(x) = \begin{cases} 0 & \text{for} \quad x \neq 0 \\ \infty & \text{for} \quad x = 0 \end{cases}$$

Such that:

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1$$

Whatever definition might be the "rigorous" one, a delta function, roughly speaking, is just a very large peak near $x = 0$ with area normed to 1:

http://en.wikipedia.org/wiki/Dirac_delta_function
http://en.wikipedia.org/wiki/Triangular_function

The latter reference is relevant too. Because it is typical that the following function, triangular as well, is supposed to converge to the delta function - instead of zero - for $n \to \infty$ and nobody has any doubt about it.

$$D_n(x) = \begin{cases} n^2.x + n & \text{if} \quad -1/n \le x \le 0 \\ n - n^2.x & \text{if} \quad 0 \le x \le +1/n \\ 0 & \quad\ \ \text{everywhere else.} \end{cases}$$

The only thing that distinguishes $g_n(x)$ from $D_n(x)$ is that the maximum of the former is shifted an infinitesimal distance $\lim_{n\to\infty} 1/(2n)$ with respect to the maximum of the latter at $x = 0$. So it's easy to see that these functions become one and the same for $n \to \infty$:

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} D_n(x) = \delta(x)$$

Note that, in this case, it's *not* even the wrong question. It's just that professors expect a wrong answer from their students. The standard answer is wrong, namely, and this one is right:

$$\lim_{n\to\infty} \left[\int_0^1 g_n(x)\, dx \right] = 1 = \int_0^1 \left[\lim_{n\to\infty} g_n(x) \right] dx$$

So again, the limit and the integration *can* be interchanged, as always it seems.

Commutativity

Definition. Let F be a function of two variables that is defined in some circular region around (a, b). The *standard* limit of F as (x, y) approaches (a, b) equals L if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that F satisfies:

$$|F(x, y) - L| < \epsilon$$

whenever the distance between (x, y) and (a, b) satisfies:

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

We will use the following notation for such limits of functions of two variables:

$$\lim_{(x,y)\to(a,b)} F(x, y) = L$$

If we replace (a, b) by infinity, then for two dimensions the definition would become slightly different, namely:

$$\lim_{|(x,y)|\to\infty} F(x, y) = L$$

If and only if for every $\epsilon > 0$ there exists $N > 0$ such that F satisfies:

$$|F(x, y) - L| < \epsilon$$

Whenever:

$$\sqrt{x^2 + y^2} > N$$

However, this case doesn't deserve much attention, because we can always write the following:

$$\lim_{|(x,y)|\to\infty} F(x, y) = \lim_{(x,y)\to(0,0)} F(1/x, 1/y)$$

Thus reducing the ∞ limit to 2-D standard. Therefore we consider the following *iterated* limit:

$$\lim_{y\to b} \left[\lim_{x\to a} F(x, y)\right] = L$$

Theorem.

$$\lim_{y\to b} \left[\lim_{x\to a} F(x, y)\right] = \lim_{x\to a} \left[\lim_{y\to b} F(x, y)\right] = \lim_{(x,y)\to(a,b)} F(x, y)$$

Proof. We split the iterated limit in two pieces:

$$\lim_{x\to a} F(x, y) = F_a(y)$$

And:

$$\lim_{y\to b} F_a(y) = L$$

Thus it becomes evident that the iterated limit is actually *defined* as follows. For every number $\epsilon_x > 0$ there is some number $\delta_x > 0$ such that:

$$|F(x, y) - F_a(y)| < \epsilon_x \quad \text{whenever} \quad 0 < |x - a| < \delta_x$$

For every number $\epsilon_y > 0$ there is some number $\delta_y > 0$ such that:

$$|F_a(y) - L| < \epsilon_y \quad \text{whenever} \quad 0 < |y - b| < \delta_y$$

Applying the triangle inequality $|a| + |b| \geq |a + b|$ gives:

$$|F(x, y) - F_a(y)| + |F_a(y) - L| \geq |F(x, y) - L| \quad \Longrightarrow$$

$$|F(x, y) - L| < \epsilon_x + \epsilon_y$$

On the other hand we have:

$$0 < |x - a| < \delta_x \quad \text{and} \quad 0 < |y - b| < \delta_y \quad \Longrightarrow$$

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \sqrt{\delta_x^2 + \delta_y^2}$$

This is exactly the definition of the above standard limit of a function of two variables if we only put:

$$\epsilon = \epsilon_y + \epsilon_y \quad \text{and} \quad \delta = \sqrt{\delta_x^2 + \delta_y^2}$$

Therefore:

$$\lim_{y \to b} \left[\lim_{x \to a} F(x,y) \right] = \lim_{(x,y) \to (a,b)} F(x,y)$$

In very much the same way we can prove that:

$$\lim_{x \to a} \left[\lim_{y \to b} F(x,y) \right] = \lim_{(x,y) \to (a,b)} F(x,y)$$

Corollaries.
1. Iterated limits simply turn out to be special cases of the standard limit in two dimensions.
2. If the standard limit does exist, then associated iterated limits do also exist. If the standard limit does not exist, then associated iterated limits do not exist as well.
3. Iterated limits always commute, if they exist. Iterated limits that "do not commute" do not exist. But the reverse is not true. Iterated limits that "commute" (by coincidence) do not necessarily exist. Mind the quotes.
4. The proof is quite standard, surprisingly simple and it doesn't make explicit use of the Axiom that *Completed infinity is not given*.
5. As far as limits are concerned, there is no need, apparently, for introducing a separate Axiom. The reason is that such an Axiom is already part of the limit definition itself.
6. It is tempting to even say that the Axiom can safely be restated as: *Infinities can only be approached through limits and nothing else.*
7. If the Axiom is relevant to science, rather than to mathematics, then the apparent commutativity of iterated limits, in scientific applications, may have tremendous consequences for the more general question what mathematics is relevant for science and what not.

Infinitesimals again

Consider again the original definition of Leibniz' Identity for two objects x, y :

$$(x = y) :\Longleftrightarrow [\ \forall P : P(x) \Longleftrightarrow P(y)\ ]$$

It has been argued in *Mathematical Identity* that for literally *all predicates* is a rather pointless idea. If you are not convinced yet, take a look at this:

$=$ 40502,28

Especially so if the objects are (real) numbers, where a binary representation is not only well known, but quite to the point as well:

$$P_k(x) \;:\Longleftrightarrow\; \text{bit } k \text{ of the binary representing the number } x \text{ is up}$$

Now if we write, instead of the original Leibniz' Identity definition, the following, what would be the difference?

$$(x \equiv y) :\Longleftrightarrow [\ \forall P \in \{P_0(x), P_1(x), P_2(x), \cdots, P_N(x), \cdots\} : P(x) \Longleftrightarrow P(y)\ ]$$

Employing infinite limits, as has been referred to in the section *Limit Dynamics*, an alternative definition would sound like this:

$$(x \equiv y) :\Longleftrightarrow \lim_{N\to\infty} [\ \forall P \in \{P_0(x), P_1(x), P_2(x), \cdots, P_N(x)\} : P(x) \Longleftrightarrow P(y)\ ]$$

Meaning nothing else than: for every number $N > 0$ there exists some number $M > 0$ such that $M > N$. It doesn't matter much if you find that this limit does not exist, because it will soon be found that the purported procedure is anyway nonsensical. But for the moment being, let's proceed. It's a matter of routine to prove that common properties of equality (reflexive, symmetric, transitive) are not different with the thus idealized finitary identity:

$$x \equiv x \quad \text{and} \quad (x \equiv y) \Longrightarrow (y \equiv x) \quad \text{and} \quad (x \equiv y) \wedge (y \equiv z) \Longrightarrow (x \equiv z)$$

Instead of $x \equiv y$ we could also have written, according to past habits: $x \overset{\infty}{=} y$. It is also clear that the binary representation is quite convenient, but not essential. The same idea can be expressed with the common decimal representation, at cost of a somewhat more complicated definition of the predicates involved.
Example. The following has given rise to many instances of an (in)famous controversy on the internet:

$$1 = 0.999999999.. \qquad \text{Is it true or false?}$$

The answer is certainly *true*. And the reason is:

$$0.99999999.. = \lim_{n\to\infty}\left[9.\frac{1}{10} + 9.\frac{1}{10^2} + 9.\frac{1}{10^3}\ ..\ + 9.\frac{1}{10^n}\right] =$$

$$\lim_{n\to\infty}\frac{9}{10}\left[1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3}\ ..\ + \frac{1}{10^{n-1}}\right] =$$

$$\lim_{n\to\infty}\frac{9}{10}\frac{1-(1/10)^n}{1-1/10} = \lim_{n\to\infty}[1-(1/10)^n] = 1$$

Somewhat surprising perhaps, but those who *object* to this result are not entirely wrong. That is because, indeed, there is some ambiguity involved with the equality definition in common mathematics. If the equality $=$ is interpreted as a Leibniz' Identity $\equiv$, then let some of the relevant predicates be defined as:

$$P_k(x) :\Longleftrightarrow \text{decimal } k \text{ is cipher } 9$$

It is clear that the following statement is *false*, more evidence is not needed:

$$P_7(1.0000000000..) \Longleftrightarrow P_7(0.999999999..)$$

So the answer to the question as a whole is certainly *false*: $1 \not\equiv 0.999999999...$
Idealization. With the above in mind, two real numbers x and y are called *equal* if and only if:

$$(x = y) \;:\Longleftrightarrow\; \lim_{\#S \to \infty} \left[\left(x \stackrel{T}{=} y \right) \vee \left(0 \stackrel{T}{<} |x-y| \stackrel{T}{<} \delta \right) \quad \text{where} \quad \delta \stackrel{S}{=} 0 \right]$$

Where $\#S$ is the number of predicates in the Smallest of the double identity aspects, called Small and Tall. Moreover, $S \subset T$ and $\#S < \#T$. Note that the term *Definition* is out of the question, because equality is considered as being defined anyway.
The equality of two real numbers now can be expressed as follows: x and y are equal if they are identical, or if the absolute value of their difference is infinitesimally small. Hence, again, the name *Infinitesimal Equality* for the second part of the definition:

$$(x = y) \;:\Longleftrightarrow\; [\, x \equiv y \,] \;\vee\; \left[\lim_{\#S \to \infty} |x-y| = 0 \right]$$

With the above example, we see that 1 and 0.999999999.. are not identical, but the two numbers are nevertheless equal, due to the *or* clause ($\vee$) and the infinitesimal equality:

$$|1 - 0.9999999999..| = \lim_{\#S \to \infty} (1/10)^{\#S} = 0$$

Despite of all controversy, a proper idealization of infinitesimals, with help of the above, seems to be rather straightforward. If the same specifications for the aspects of the mathematical identities involved are adopted, then we have the following
Idealization. A real number dx is infinitely small, *infinitesimal* for short, if the following is valid:

$$\lim_{\#S \to \infty} \left[0 \stackrel{T}{<} |dx| \stackrel{T}{<} \delta \quad \text{where} \quad \delta \stackrel{S}{=} 0 \right]$$

According to our definition of the Leibniz' Identity, we could boldly write now:

$$(x = y) \;:\Longleftrightarrow\; (x \equiv y) \;\vee\; [\; (0 < |x-y| < \delta) \quad \text{where} \quad \delta \equiv 0 \;]$$

Assuming that, with proper idealization of the identity, there is also a proper idealization of the non-identities $<$ and $>$. But here we've lost all control! With the Tall aspect we have:

$$(x \equiv y) \;:\Longleftrightarrow\; \lim_{\#T \to \infty} [\; \forall P \in T : P(x) \Longleftrightarrow P(y) \;]$$

With the Small aspect we have, on the other hand:

$$(\delta \equiv 0) \;:\Longleftrightarrow\; \lim_{\#S\to\infty} [\, \forall P \in S : P(\delta) \Longleftrightarrow P(0) \,]$$

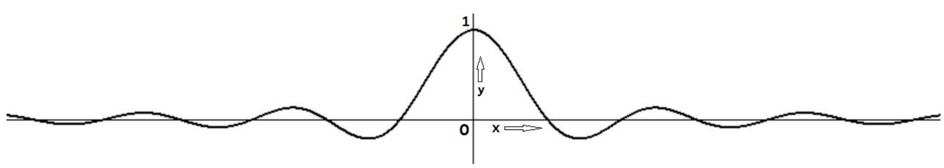
But with a notation like $(x \equiv y)$ and $(\delta \equiv 0)$ it is not at all clear anymore whether the limits - if they may be assumed to exist - are meant to be with the small or with the tall aspect. With other words: if it is assumed that we can finish the limits, then the Double Identity, which is essential for the proper understanding of infinitesimals, is completely lost.

- Defining infinitesimals within a mathematical framework with completed limits is extremely hard, if not impossible

Brouwer's Continuity Theorem

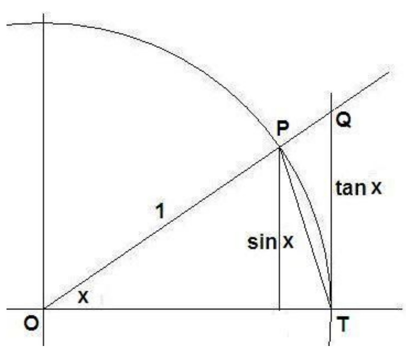
Quite useful in physics applications (e.g. optics) is the so-called *sinc* function, which is commonly defined as follows:

$$\text{sinc}(x) = \begin{cases} \sin(x)/x & \text{for} \quad x \neq 0 \\ 1 & \text{for} \quad x = 0 \end{cases}$$



The function is continuous (and differentiable) everywhere. Especially it is continuous at $x = 0$. Proof of the latter rests on a well known limit:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 = \text{sinc}(0)$$



Part of the unit circle is shown. From the picture it is clear that: triangle area (OTP) < circle sector (OTP) < triangle area (OTQ). Algebraically this is translated into:

$$\sin(x)/2 < x/2 < \tan(x)/2$$

$$\Longleftrightarrow \quad 1 < \frac{x}{\sin(x)} < \frac{1}{\cos(x)} \quad \Longleftrightarrow \quad \cos(x) < \frac{\sin(x)}{x} < 1$$

For $x \to 0$ not only the right hand side is 1 but the left hand side becomes 1 as well: $\lim_{x\to 0}\cos(x) = 1$. The sinc function thus becomes trapped between two values equal to 1. Therefore it can itself be nothing else than 1. Mind, however, that the value $x = 0$ is never reached. This brings us to the following alternative definition. Oh yes, among numerous other possibilities, but this one will do:

$$\operatorname{sinc}(x) = \begin{cases} \sin(x)/x & for \quad x \neq 0 \\ 2 & for \quad x = 0 \end{cases}$$

Would such a definition be possible? In a mathematics with completed infinities the answer is certainly *yes*. Because we *have* all those real numbers in the first place, each of them as a completed entity, i.e. each of them as a finished limit - most of the time, because most real numbers are irrationals. Given these finished limits, continuity is defined by just another limit.

But here is exactly the sting! When establishing the continuity of the sinc function, the former limits - invariably associated with real numbers x as the idealized errors of them - are intermixed with the latter limit, the one namely for establishing continuity. Therefore, what's really happening here is that the limits of real number definitions and the limit defining continuity are intermixed. Or should we say: messing up. Whatever. What we actually have here are *iterated limits*. We have seen that the only way to be consistent with *Infinitum Actu Non Datur* is to "complete" both parts of an iterated limit at the same time. Thus we have the definition of a real and continuity in one big sweep and not one after the other. There is another way of looking at it. Remember the Approximate Equality definition:

$$(x \approx y) \;:\Longleftrightarrow\; \left[\left(x \overset{T}{=} y \right) \vee \left(0 \overset{T}{<} |x-y| \overset{T}{<} \delta \right) \right] \qquad \text{where} \quad \delta \overset{S}{=} 0$$

To be compared with the definition of continuity. A function f is *continuous* at some number a if:

$$\lim_{x \to a} f(x) = f(a)$$

Meaning that if for every number $\epsilon > 0$ there is some number $\delta > 0$ such that:

$$|f(x) - f(a)| < \epsilon \quad \text{whenever} \quad |x-a| < \delta$$

Even more striking than the translation of the limit definition in material terms is the translation of the continuity definition, because mathematical identity is present here instead of absent:

- $\left[f(x) \overset{T}{=} f(a) \right]$ whenever $\left[x \overset{T}{=} a \right]$
- $\left[0 \overset{T}{<} |f(x) - f(a)| \overset{T}{<} \epsilon \quad \text{where} \quad \epsilon \overset{S}{=} 0 \right]$
  whenever $\left[0 \overset{T}{<} |x-a| \overset{T}{<} \delta \quad \text{where} \quad \delta \overset{S}{=} 0 \right]$

The first bullet defines what a function in (material) mathematics really is, the second bullet defines two Infinitesimal Equalities. Together they form two *Approximate Equalities*. Therefore, upon careful inspection of the Continuity definition the following extremely simple pattern is observed:

$$x \approx a \quad \Longrightarrow \quad f(x) \approx f(a)$$

Taking the limit - involving *idealization* of continuity:

$$x = a \quad \Longrightarrow \quad f(x) = f(a)$$

Thus we see that, with material mathematics, *equality and continuity merge into one another*.

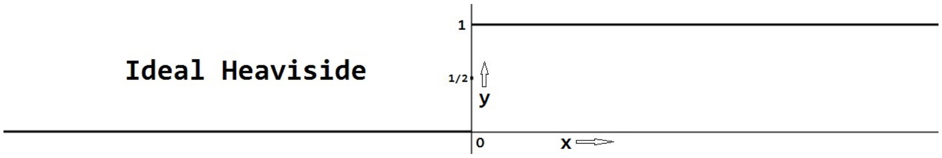
When considering Constructivism and Intuitionism, the discussion has been severely hampered by the fact that intuitionistic logic - especially its problems with the *Law of excluded middle* - has been over-emphasized, ever since the infamous debate between Hilbert and Brouwer took place. As a physicist, I find another achievement of Intuitionism far more intriguing than its contributions to Mathematical Logic.

After some search, we arrive at *Brouwer's Continuity Theorem*, boldly stating that: *Every total real function is continuous.* Somewhat more elaborate: *Any function which is defined everywhere at an interval of real numbers is also continuous at the same interval.* With other words: For real valued functions, being defined is very much the same as being continuous.

Why is this so interesting? Well, because of the fact that, in e.g. physics, "neat", everywhere continuous functions are employed most of the time. And Brouwer's Continuity Theorem might provide evidence why this simply must be so. Maybe he intuitionist's continuum is indeed more resemblant to the real-world continuum (in physics) than the classical continuum has ever been. This is the main motivation for my "pink intuitionism" (in honour of Torkel Franzén).

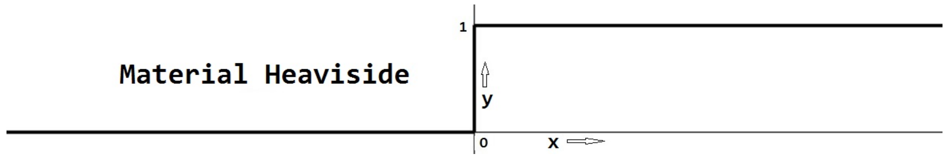
Example. Quite useful in physics applications (e.g. electronics) is the so-called *Heaviside* function, which is commonly defined as follows:

$$H(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \frac{1}{2} & \text{for} \quad x = 0 \\ 1 & \text{for} \quad x > 0 \end{cases}$$



According to Brouwer's Continuity Theorem, this is utterly impossible. Instead, the Heaviside function is continuous, *hence a function*, for $x < 0$ and for $x > 0$. It is discontinuous, *hence not a function*, for $x = 0$. As follows:

$$H(x) = \begin{cases} y = 0 & \text{for} \quad x \le 0 \\ 0 \le y \le 1 & \text{for} \quad x = 0 \\ y = 1 & \text{for} \quad x \ge 0 \end{cases}$$



With other words, the "material" Heaviside function for $x = 0$ can have any value between 0 and 1 and therefore is not a function in the strict sense of the word at that place (the term *generalized function* is coined up):

`http://mathworld.wolfram.com/HeavisideStepFunction.html`

To end where we started, this is the content of Brouwer's Continuity Theorem - idealized version - for our sinc function:

$$x = 0 \quad \Longrightarrow \quad \text{sinc}(x) = 1$$

Hence we must conclude that a definition like $\text{sinc}(0) = 2$, though "possible" in traditional theory, is utterly *nonsensical* in practice.
Theorem. It is impossible to have a definition for the value of the sinc function at $x = 0$ other than $\text{sinc}(0) = 1$. Therefore we shall simply define it as follows. All the rest is a mere consequence of this definition:

$$\text{sinc}(x) = \frac{\sin(x)}{x}$$

Square Root of Two

One problem with classical mathematics is that it makes a sharp and rather unrealistic distinction between equality of two real numbers and continuity of real valued functions. In physical (and material mathematical) practice these two things are essentially the same. Suppose, for example, that you simply don't "have" any irrational numbers at your disposal; therefore you have to define $\sqrt{2}$ (square root of two), all by yourself. Such a thing can be accomplished, with an algorithm, as follows. Disclaimer: this algorithm is not by far the most efficient way to calculate $\sqrt{2}$, because it roughly gives only one bit with each iteration. The algorithm is coded as a (Delphi) Pascal program:

```
program wortel;
{
  Square Root of Two
}
const { epsilon }
  eps : double = 1.e-12;
var
  x,h,y,T,B : double;
begin
```

```
{ (4/3)^2 = 16/9 < 2
  (3/2)^2 =  9/4 > 2
  So initialize with:
  4/3 < sqrt(2) < 3/2 }
  T := 3/2; B := 4/3;
  x := (B+T)/2;
  h := (T-B)/2;
  y := x*x;
  while true do
  begin
  { Binary search }
    if y > 2 then
    begin
      T := x ; x := x - h;
    end;
    if y < 2 then
    begin
      B := x ; x := x + h;
    end;
    y := x*x;
    Writeln('sqrt(2) =',x,' ; 2 =',y);
  { Continuity requirement }
    if T-B < eps then Break;
    h := h/2; { interval }
  end;
{ Comparison with standard value }
  Writeln('         ',sqrt(2));
  Writeln('epsilon =',eps,' > T - B');
  Writeln('B =',B,' < x <',T,' = T');
  Writeln('|x - a| = |x - sqrt(2)| < T - B =',T-B);
  Writeln('  for |f(x) - f(a)| = |x^2 - 2| =',abs(y-2));
  Writeln('hence |x - sqrt(2)| = |x^2 - 2|/(2.x) =',abs(y-2)/(2*x));
end.
```

Output:

```
sqrt(2) = 1.33333333333333E+0000 ; 2 = 1.77777777777778E+0000
sqrt(2) = 1.37500000000000E+0000 ; 2 = 1.89062500000000E+0000
sqrt(2) = 1.39583333333333E+0000 ; 2 = 1.94835069444444E+0000
sqrt(2) = 1.40625000000000E+0000 ; 2 = 1.97753906250000E+0000
sqrt(2) = 1.41145833333333E+0000 ; 2 = 1.99221462673611E+0000
sqrt(2) = 1.41406250000000E+0000 ; 2 = 1.99957275390625E+0000
sqrt(2) = 1.41536458333333E+0000 ; 2 = 2.00325690375434E+0000
sqrt(2) = 1.41471354166667E+0000 ; 2 = 2.00141440497504E+0000
sqrt(2) = 1.41438802083333E+0000 ; 2 = 2.00049347347683E+0000
sqrt(2) = 1.41422526041667E+0000 ; 2 = 2.00003308720059E+0000
sqrt(2) = 1.41414388020833E+0000 ; 2 = 1.99980291393068E+0000
```

```
sqrt(2) = 1.41418457031250E+0000 ; 2 = 1.99991799890995E+0000
sqrt(2) = 1.41420491536458E+0000 ; 2 = 1.99997554264135E+0000
sqrt(2) = 1.41421508789062E+0000 ; 2 = 2.00000431481749E+0000
sqrt(2) = 1.41421000162760E+0000 ; 2 = 1.99998992870355E+0000
sqrt(2) = 1.41421254475911E+0000 ; 2 = 1.99999712175405E+0000
sqrt(2) = 1.41421381632487E+0000 ; 2 = 2.00000071828415E+0000
sqrt(2) = 1.41421318054199E+0000 ; 2 = 1.99999892001870E+0000
sqrt(2) = 1.41421349843343E+0000 ; 2 = 1.99999981915132E+0000
sqrt(2) = 1.41421365737915E+0000 ; 2 = 2.00000026871771E+0000
sqrt(2) = 1.41421357790629E+0000 ; 2 = 2.00000004393451E+0000
sqrt(2) = 1.41421353816986E+0000 ; 2 = 1.99999993154292E+0000
sqrt(2) = 1.41421355803808E+0000 ; 2 = 1.99999998773871E+0000
sqrt(2) = 1.41421356797218E+0000 ; 2 = 2.00000001583661E+0000
sqrt(2) = 1.41421356300513E+0000 ; 2 = 2.00000000178766E+0000
sqrt(2) = 1.41421356052160E+0000 ; 2 = 1.99999999476319E+0000
sqrt(2) = 1.41421356176337E+0000 ; 2 = 1.99999999827543E+0000
sqrt(2) = 1.41421356238425E+0000 ; 2 = 2.00000000003154E+0000
sqrt(2) = 1.41421356207381E+0000 ; 2 = 1.99999999915348E+0000
sqrt(2) = 1.41421356222903E+0000 ; 2 = 1.99999999959251E+0000
sqrt(2) = 1.41421356230664E+0000 ; 2 = 1.99999999981203E+0000
sqrt(2) = 1.41421356234544E+0000 ; 2 = 1.99999999992179E+0000
sqrt(2) = 1.41421356236485E+0000 ; 2 = 1.99999999997667E+0000
sqrt(2) = 1.41421356237455E+0000 ; 2 = 2.00000000000411E+0000
sqrt(2) = 1.41421356236970E+0000 ; 2 = 1.99999999999039E+0000
sqrt(2) = 1.41421356237212E+0000 ; 2 = 1.99999999999725E+0000
sqrt(2) = 1.41421356237333E+0000 ; 2 = 2.00000000000068E+0000
sqrt(2) = 1.41421356237273E+0000 ; 2 = 1.99999999999896E+0000
sqrt(2) = 1.41421356237303E+0000 ; 2 = 1.99999999999982E+0000
          1.41421356237310E+0000
epsilon = 1.00000000000000E-0012 > T - B
B = 1.41421356237273E+0000 < x < 1.41421356237333E+0000 = T
|x - a| = |x - sqrt(2)| < T - B = 6.06403816050261E-0013
  for |f(x) - f(a)| = |x^2 - 2| = 1.82520665248376E-0013
hence |x - sqrt(2)| = |x^2 - 2|/(2.x) = 6.45308000519061E-0014
```

Thus, given any small number called ϵ, by employing binary search, we find a number x such that $B < x < T$ (between Bottom and Top), where $T - B < \epsilon$. Thus it is certain that the number x is close to the square root of 2 with a predefined error: $|x - \sqrt{2}| < \epsilon$. The algorithm is steered by two comparisons: $x^2 > 2$ or $x^2 < 2$. There is an error involved with $|x^2 - 2|$ as well, let's call it δ. One might ask for a better error estimate for x. With other words: what is the number ϵ in $|x^2 - 2| < \delta \;\Rightarrow\; |x - \sqrt{2}| < \epsilon$? Reasoning goes as follows: $|x^2 - 2| = |x + \sqrt{2}||x - \sqrt{2}| \approx 2x\epsilon = \delta$. Gives $\epsilon \approx \delta/(2x)$. Here $f'(x) = 2x$ is not at all by coincidence the derivative of $f(x) = x^2$. Indeed derivatives *are* routinely employed for error processing, as in $y = x^2 \Rightarrow dy/dx = 2.x \Rightarrow dx = dy/(2x)$ where $\epsilon = dx$ and $\delta = dy$.

It is concluded that finding $\sqrt{2}$ is quite intimately related to a procedure of the following form: find a real number $\delta > 0$ such that for any real $\epsilon > 0$: $|f(x) - f(a)| < \delta \;\Rightarrow\; |x - a| < \epsilon$. Here x is the unknown number (as usual), $a = \sqrt{2}$, $f(x) = x^2$ and $f(a) = 2$. Even superficial inspection reveals that we have re-discovered here the classical definition of continuity. But it is just the other way around. To be precise, it is the classical definition for the *inverse function* of the parabola $f(x) = x^2$, which is, of course, the yet "undefined" square root function $f^{-1}(x) = \sqrt{x}$. With other words: $\sqrt{2}$ can be found only because there exists a function $f(x) = x^2$ on the reals which has an inverse and it is continuous. Thus the number x is forced to assume a certain value for $f(x) = 2$. The gist of Brouwer's Continuity Theorem is that the very definition of a real number, essentially, cannot be distinguished from the continuity of associated real valued functions. Real numbers and real functions have to be idealized, if not defined, in one big sweep, instead of the former by e.g. Dedekind cuts, and the latter by some other means, at some time later on.