Material Equality

With our modern programming languages, there is usually a sharp distinction between numerical data types, the most important being *integer* and *real*. Quite important because integer and real respectively represent the discrete and the continuous in nature. Therefore, what we shall do first is build an abstraction for the integers, or rather: the naturals. An abstract non idealized theory for the real numbers is our next topic. Once these two subjects have been covered, negative numbers, rationals and complex numbers, to mention a few topics, are not really an issue anymore.

Mathematical Identity

There exists a *Theory of Identity* in mathematical logic. I've encountered it for the first time in *Principia Mathematica* by Alfred North Whitehead and Bertrand Russell (1910):

`http://en.wikipedia.org/wiki/Principia_Mathematica`

Quote: "This definition states that x and y are to be called identical when every predicative function satisfied by x is also satisfied by y". Many contemporary philosophers call the principle which expresses this view "Leibniz' Law". One particularly explicit statement can be found in *Introduction to Logic and to the Methodology of Deductive Sciences* by Alfred Tarski:

`http://en.wikipedia.org/wiki/Alfred_Tarski`

In chapter III, *On the Theory of Identity*, it is read that "Among logical laws which involve the concept of identity, the most fundamental is the following:
$x = y$ if, and only if, x and y have every property in common.
This law was first stated by Leibniz (although in somewhat different terms)."
Tarski does not provide a reference to the place where, according to him, Leibniz stated that law. Further refinements can be found on the internet:

`http://en.wikipedia.org/wiki/Identity_of_indiscernibles`

There exist two formulations, which seem to be very much alike, but not quite, so let's be careful:

- 1. The indiscernibility of identicals
  For any x and y, if x is identical to y, then x and y have all the same properties.
  $$\forall x \forall y \, [x = y \rightarrow \forall P \, \{P(x) \leftrightarrow P(y)\}]$$
- 2. The identity of indiscernibles
  For any x and y, if x and y have all the same properties, then x is identical to y.
  $$\forall x \forall y \, [\forall P \, \{P(x) \leftrightarrow P(y)\} \rightarrow x = y]$$

For the purpose of our thesis, it is sufficient to stick to the original definition, as given with the *Theory of Identity* by Tarski / Russell and Whitehead:

$$(x = y) :\Longleftrightarrow [\ \forall P : P(x) \Longleftrightarrow P(y)\ ]$$

Where $:\Longleftrightarrow$ means: logically equivalent by definition. Let's try something with that definition. *Every* property in common, they say. We take that quite literally and have, for example:

$$P(x) :\Longleftrightarrow (x \, \text{is on the left of the}" = " \, \text{sign})$$

With this property in mind, consider the expression:

$$1 = 1$$

Then we see that the 1 on the right in $1 = 1$ is not on the left, hence the property $P(1)$ as defined does not hold for that one. Consequently: $1 \neq 1$. We have run into a Paradox. Oh, you should say, but self-referential properties are of course not allowed. Sure, I am the last one to disagree with you. This highly artificial example stresses an important point, though:

- With Leibniz's Law, almost any but *not all* properties are in common

The numerosity of these (not self-referential) properties can still be infinite, though. So, in concordance with *Infinitum Actu Non Datur*, we shall have the additional requirement that these Leibniz Properties are *finite* in number. Let I_N be a finite set of properties P_k and let's call I_N the *aspect* or *scope* of the equality:

$$I_N := \{\ P_0(x), P_1(x), P_2(x), ..., P_N(x)\ \}$$

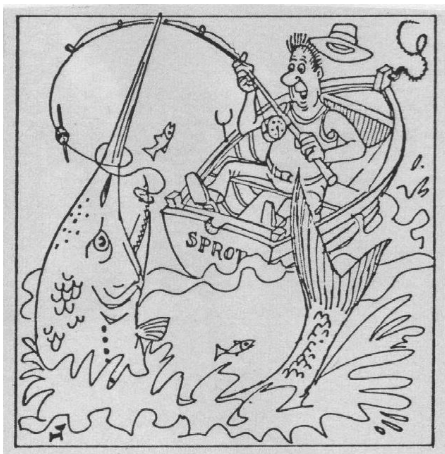
Then $(x = y)$ shall be pronounced as x **is equal to** y **with respect to** I_N, and may optionally be written as:

$$(x \stackrel{N}{=} y) :\Longleftrightarrow [\ \forall P_k \in I_N : P_k(x) \Longleftrightarrow P_k(y)\ ]$$

It's a matter of routine to prove that common properties of equality (reflexive, symmetric, transitive) are not different with the above modified definition:

$$x \stackrel{N}{=} x \quad \text{and} \quad (x \stackrel{N}{=} y) \Longrightarrow (y \stackrel{N}{=} x) \quad \text{and} \quad (x \stackrel{N}{=} y) \wedge (y \stackrel{N}{=} z) \Longrightarrow (x \stackrel{N}{=} z)$$

Up to now, we have not been very clear about what sort of properties one should have in mind, when comparing object x with object y in some respect. This topic will be covered further in the next few sections, for two special cases: natural numbers and real numbers. But a nice example of the non-triviality of (non)identity can be given here and now. Suppose x and y are pictures, like in well known puzzles for children, when they say "find the differences":



x

y

Instead of a single index (k), a double index (i,j) may be preferred here for the properties P:

$$P(i,j) \;:\Longleftrightarrow\; \text{"pixel at position } (i,j) \text{ is black"}$$

Herewith we have investigated a few facts of *being identical* in the finite domain. It is a *materialization* of an ideal identity and may be called a *material identity*:

$$(x \overset{N}{=} y) :\Longleftrightarrow [\, \forall P_k \in I_N : P_k(x) \Longleftrightarrow P_k(y) \,]$$

Facts of *being identical* in an absolute, *idealized* sense have been investigated elsewhere. With Leibniz' definition it is clear that, indeed, infinity is involved:

$$(x = y) :\Longleftrightarrow [\, \forall P : P(x) \Longleftrightarrow P(y) \,]$$

It may be not difficult to accept that being equal in some finite respect is the obvious materialization of being equal in any respect, ipse est: absolute identity. The other way around - idealization of a material identity to the common ideal identity - is somewhat more tricky. One could be tempted to say that the ideal identity is the *limit* of a material identity, which would formally be as follows:

$$(x = y) :\Longleftrightarrow \lim_{N\to\infty} [\, \forall P_k \in I_N : P_k(x) \Longleftrightarrow P_k(y) \,]$$

Then $(x = y)$ would be equivalent with $(x \overset{\infty}{=} y)$. There is a minor problem, however: it is not at all obvious how the formal definition of a limit could be applicable in this case.

Keeping a Tally

This is what Leopold Kronecker said (1886): "Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk" (the whole numbers have been made by God, all the rest is work of humans) Though we feel great sympathy for Kronecker's constructive stand, we do not agree with his statement that the naturals would be the work of a supernatural being.
At the time I was a regular visitor of the local pub, the barkeeper used to keep a record of the beer mugs on my account by scribbling strokes '|' on a piece of paper, as follows: | , || , ||| , |||| :

`http://en.wikipedia.org/wiki/Tally_marks`

Such a procedure is called "turven" in Dutch - "score" or "keep a tally" in English. Paper has not always been the favorite medium for keeping a tally:

`http://en.wikipedia.org/wiki/Tally_sticks`
`http://nl.wikipedia.org/wiki/Kerfstok`

One reason why the barkeeper's method is part of this thesis is that there exist a constructivist's method which is quite similar to it; I've looked it up in the notes of an intuitionist mathematician (W. Peremans, Eindhoven, 1972, one of my professors in mathematics at the Eindhoven University of Technology) The gist of the argument in these notes is that it is possible to *define* the natural numbers rigorously and constructively in a manner very much similar to the barkeeper's method. We shall not dwell on nasty formalities and accept this as a mere fact. Intuitionism without the trouble; it has been called *pink intuitionism* by Torkel Franzén, an internet legend:

`http://en.wikipedia.org/wiki/Torkel_Franz%C3%A9n`

Thus it is supposed in the sequel that everybody understands the art of creating tally marks and knows how to do counting with them:

```
|  ||  |||  |||| |||||  ||||||  |||||||  ||||||||  |||||||||  ||||||||||
```

Time for a little game now. But first we are going to create sort of a standard set of natural numbers in the following manner. Start with the number (one): | . Create the next number (two) by concatenating one to itself: | | $\rightarrow$ || . Create the next number (four) by concatenating two to itself: || || $\rightarrow$ |||| . In general, create the next number by concatenating the previous number to itself: $x\ x \rightarrow xx$, where x is the previous natural. This provides us with the following set of numbers (powers of two):

```
|  ||  ||||  ||||||||  ||||||||||||||||  ||||||||||||||||||||||||||||||||
```

Where it is decided that we stop after the fifth concatenation. As a matter of fact, our standard set will be employed in reverse order, largest numbers first:

```
||||||||||||||||||||||||||||||||  ||||||||||||||||  ||||||||  ||||  ||  |
```

For reasons that will become clear later on, we shall call the above powers of two set a *calibration set*:

`http://en.wikipedia.org/wiki/Calibration`

Another ability is needed for successful completion of the game, namely the ability to decide if one natural is *smaller than or equal to* another natural. There is no need to be able to count the marks in a tally for that purpose. A comparison (by means of a bijection) between the two naturals shall reveal if there are less marks in one of the two or not:

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | | | | | | | | | |

In set theoretic terms, we must be able to decide if one natural is a subset - or rather a *substring* - of the other, Yes or No:

$$||||||||||||||||||||||||| \subset ||||||||||||||||||||||||||||||||||||| \quad ? \quad \text{Yes!}$$

$$|||||||||||||||||||||||||||||| \subset ||||||||||||||| \quad ? \quad \text{No!}$$

$$||| \subset ||| \quad ? \quad \text{Yes!}$$

So here comes the game. Think of a certain number between 0 and 64; please don't tell anybody what it is. Our task is to guess that number of yours in as few steps as possible. (Apologies if you had a different number in mind as has been assumed below :-)
I will start with the largest number in our calibration set. Then comes my first question: is this number (32) a substring of your secret number?

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |

I hear that your first answer is: No (0) !
So let's proceed with the second number in our calibration set. Then comes the next question: is the number (16) part of your number?

| | | | | | | | | | | | | | | |

It is heard that your second answer is: Yes (1) !
We shall *concatenate* 16 with the third number in our calibration set (8). Next question: is this number $(16 + 8 = 24)$ a subset of yours?

| | | | | | | | | | | | | | | | | | | | | | | |

Your answer: Yes (1) !
We shall concatenate 24 with the fourth largest natural in our calibration set (4). Next question: is this number $(16 + 8 + 4 = 28)$ a substring of yours?

| | | | | | | | | | | | | | | | | | | | | | | | | | | |

Your answer: No (0) !
So forget that last concatenation. Instead, we shall concatenate 24 with the next largest natural in our calibration set (2). Question: is this number $(16+8+2 = 26)$ smaller than or equal to yours?

```
||||||||||||||||||||||||||
```

Your answer: No (0) !
So forget that last concatenation. Instead, I will concatenate 24 with the last (smallest) natural in our calibration set (1). Last question: is this $(16 + 8 + 1 = 25)$ part of your number?

```
|||||||||||||||||||||||||
```

Your last answer: Yes (1) ! So the number you had in mind *must be* 25 ! Now that we know the number you were supposed to have in mind, let's summarize the search for the right answer - in six steps - once again (where <= is the same as $\subset$ and spaces for clarity):

```
|||||||||||||||||||||||||||||||| <=
|||||||||||||||||||||||||         ? : 0

||||||||||||||||                 <=
|||||||||||||||||||||||||         ? : 1

|||||||||||||||| ||||||||        <=
|||||||||||||||| |||||||||        ? : 1

|||||||||||||||| |||||||| ||||   <=
|||||||||||||||| |||||||| |       ? : 0

|||||||||||||||| |||||||| ||     <=
|||||||||||||||| |||||||| |       ? : 0

|||||||||||||||| |||||||| |      <=
|||||||||||||||| |||||||| |       ? : 1
```

Corollaries.
1. The following answers have been given, in order: 0 1 1 0 0 1. This is - not at all by coincidence - the *binary representation* of the hitherto unknown number 25 that you were supposed to have in mind.
2. Leading zeroes are part of the game; they cannot be discarded a priori, but only a posteriori: as soon as the game is over.
3. The *scope* of the calibrated numbers is limited. In our game, it corresponds with six binary digits (bits). Meaning that numbers between 0 and 64 can be observed, but no more.
4. If the restriction is removed that the number you have in mind must be between 0 and 64 - while keeping the scope of the calibration set as it is - then any number greater than (or equal to) 63 will result in 111111 (overflow).

Natural Counting

Most parents shall have some experience with teaching their children the art of counting. And nobody will deny that there exist large individual differences between one child or another. Personally, I find children who are not exhibiting so much "intelligence" from their own the more interesting. Maybe because a "not so smart child" mimics more or less the difficulties we would encounter with trying to learn a stupid machine how to count any elements in a set - how would a computer or a robot perform on "cardinal numbers"? But, in order not to embarrass anybody, let me give that child of ours a fictive name. Let's call him Tommy, but assume he is not *that* much of a "deaf, dumb and blind kid":

`http://nl.wikipedia.org/wiki/Tommy_(rockopera)`

When teaching our little Tommy how to count, we started with a few apples in a dish. And we asked him: how many apples are there in that dish? Though Tommy knew some of the counting words (ordinals) - one, two, three, four, five - he couldn't apply them to that set of apples (cardinals). This remained so for a while. While carefully observing the child, however, it gradually became clear that he always made the same mistake. One way or another, he wasn't able to count each apple *only once*. So he attached more than one counting word to each apple. And he kept running in circles, until he was bored. "Ten, mama!" Obviously, the most important idea to be grasped here is that the elements in a set have to be *distinguished* from each other. If our little Tommy doesn't effectively understand how, or why, to do this, then he will go on and on, until he becomes bored eventually or - if you are lucky - until his limited set of ordinal numbers becomes exhausted. As a rule, the latter happened rather quickly. Indeed, while looking into that dish, all apples are looking the same. If the child is not allowed to mark each of the apples, with a dirty wet finger for example, then he has no means to distinguish one apple from another and he will count some of them more than once. While this seems not to be much of a problem for a dish with five apples, try to present a dish with a hundred apples or so to any adult person and ask him / her to count them, just by looking.
The keyword here is *marking*. But, instead of the child's dirty wet finger, there is another, neater solution that can be thought of. After a couple of fruitless attempts, we did the following.
Tommy was forced to take an apple *out* of the dish. And at the same time say a counting word. Then the miracle happened: after the last apple was taken out, he stopped counting. The Halting Problem was finally solved! With every *act* of the kind, let it shout the next ordinal number. Then, instead of the endless loops in the past, suddenly a stopping criterion seems to be present. And, as soon as the dish is empty, there is no reason to count any further. A few pictures say more than a thousand words.

Examine now Tommy's counting process in detail. Taking apples out of the dish does actually *destroy* the set you want to determine the cardinality of. Even more general, it is *impossible* to count a set without disturbing it, one way or another. Even if you count apples by just taking a good look, you have to shoot photons at them, since you can't see them without light. Apart from this, you have to remember which one has been enumerated already. So you have to put *marks* upon the apples, at least in your mind. But as soon as an element is marked, then it isn't the "same" element anymore. The elements have been changed while counting them. Or, which is basically the same: they belong to another set.
All this should'nt be surprising, since counting is a physical process, sort of a measurement, in its simplest form. According to quantum mechanics (and common experience confirms it) any form of measurement implies a disturbance. We conclude that, in the process of counting, at least two sets are involved:

- the set of apples which is still to be counted ("past")
- the set of apples which has already been counted ("future")

The "past" set is destroyed, while the "future" set is created. A child fails to comprehend how to assign a number to the elements in a set. I went through Tommy's school-books and, after all, I wasn't much surprised. According to the teaching methods in those books, the children learn the order of the counting words. Sure. And they learn when two sets have an equal number of elements. But perhaps the more difficult problem is: how to assign a counting word to the number of elements in a set. Mathematically speaking: what is the relationship between ordinals and cardinals? In a sense that can be made clear to a little child? The picture is completed by *Keeping a Tally* of the counting:

$|$ $\quad$ $||$ $\quad$ $|||$ $\quad$ $||||$ $\quad$ $|||||$

Teaching methods in our schools are related to "modern" mathematics ("New Math" - sic: from the previous century), with sets that "exist" forever and do not "change". Therefore I'm pretty sure that the above practice does not belong to the standard equipment for teaching numbers; the dynamics of this is contradictory to the statics of set theory based mathematics.

Binary Balance

An important source of real numbers is the physics experiment, a *measurement*. Therefore, before launching any *theory* about the reals, I think it's good to have some *practical* experience, with some realistic physical equipment. The most common measurement probably would be measuring a length. The concept of length, however, is very much related to pure geometry, which is branch of mathematics. We all know that non-measurable numbers have emerged in geometry, meaning that the concept of length may be too well known for our purpose already. We don't want the reader to think of anything mathematical in the first place. Our measurement should be sufficiently *abstract* with respect to mathematics. Yet it must be sufficiently simple. And reasonably accurate. How about weighting? An example may be how to determine the weight of a tomato. Yes, it's not so difficult to devise a simple balance (: Google up "How to Make a Homemade Weighing Scale"). I've done it myself with a few pieces of wood, a mounting strip, a paper clip, two paper cups, two (equal) pieces of rope and two little pegs. It's not so difficult either to devise a weighting scale, with calibrated weights. Read about the "Basic calibration process" in:

`http://en.wikipedia.org/wiki/Calibration`

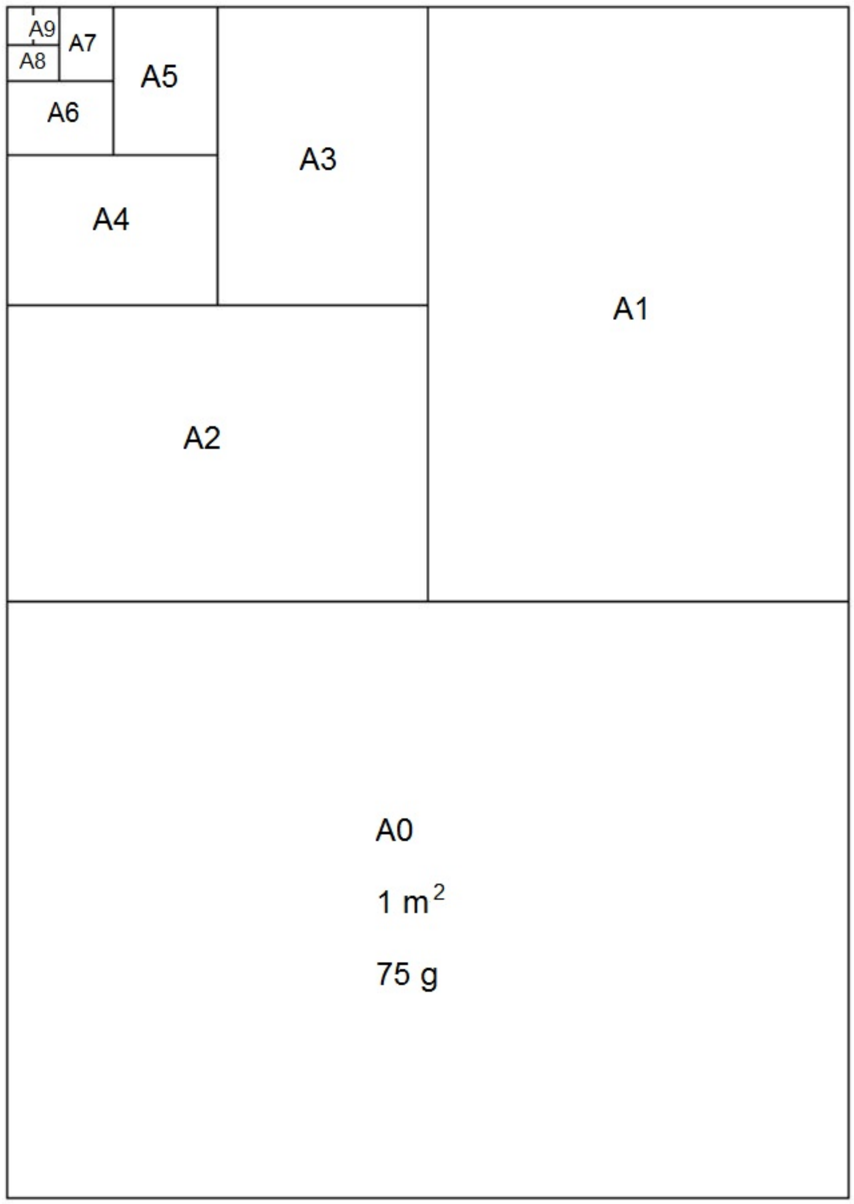
Mind that our weighting experiments are meant for revealing certain properties of the real numbers, as they emerge with measurements. So we don't have to conform to all kind of measurement standards. There is no need to adopt the kilogram as a unit, for example. Our unit of weight really can be anything. Another convention that we do not need to adopt is the decimal number system; such as with decimal weights, "suitable for general laboratory, commercial, and educational use":

```
1kg , 500 g , 2 x 200 g , 100 g , 50 g , 2 x 20 g , 10 g , 5 g ,
2 x 2 g, 1 g , 500mg , 2 x 200mg , 100mg , 50mg , 2 x 20mg , 10mg
```

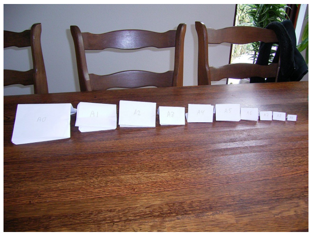
Decimal calibration is not what we are looking for indeed. The reason is simple: it's rather bothersome to manufacture these weights at home. We would like to suggest a much more feasible alternative: printing *paper*.

`http://en.wikipedia.org/wiki/Paper_size`

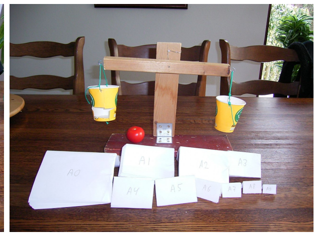
Paper, here in Europe at least, comes standard as $A4$. The weight is usually known: $75g/m^2$ is read here for common printer paper. Sixteen of these $A4$ form an $A0$ size paper, with area $= 1m^2$ (sic, why would that be?) and hence a weight of $75g$. Which by the way is not an unreasonable weight to start with. Eight $A4$ form an $A1$ size, Four $A4$ make an $A2$. Two $A4$ make an $A3$. But by far the greatest advantage is that it's very easy to fold one $A4$ and make two $A5$ of it, fold one $A5$ and make two $A6$ of it, fold one $A6$ and make two $A7$ of it, fold one $A7$ and make two $A8$ of it, fold one $A8$ and make two $A9$ of it (we only need one of those two). Okay, that seems to be enough for our purpose. What has been accomplished now is that we have a collection of ten "calibrated" weights, non-decimal, *binary* weights to be precise. A picture says more than a thousand words:

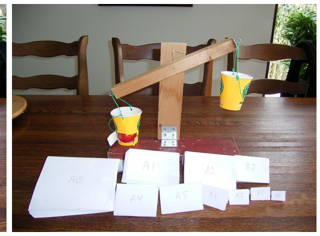


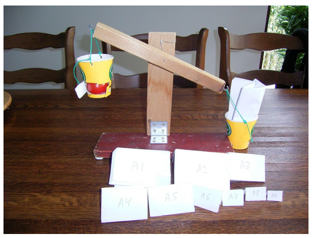
Meanwhile, we could have built our primitive balance. We take a tomato. And just start the weighting process. The tomato is put into the left cup. The paper weights are put into the right cup. In the beginning, all paper weights are on the table. We start with the heaviest ($A0$) weight and then continue with putting the lighter weights into the right cup, one by one, in this order: $A1$, $A2$, $A3$, $A4$, $A5$, $A6$, $A7$, $A8$, $A9$. See photographs on the next page:



A0 A1 A2 A3 A4 A5 A6 A7 A8 A9

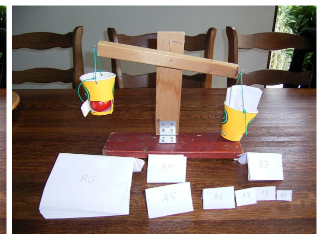






A0

A1



A1 A2

A1 A3

A1 A4

A1 A4 A5

A1 A4 A5 A6

A1 A4 A5 A7

A1 A4 A5 A8

A1 A4 A5 A9

A1 A4 A5

A1 A4 A5

If the left cup is closer to the ground then that means the right cup is too light; so we leave the last paper weight in that cup. If the right cup is closer to the ground then that means the cup is too heavy; so we take the last paper weight out and put it on the table again. If the balance is in equilibrium then we usually decide to stop. Photograph (9) is repeated for comparison with photograph (14): both show the balance "in equilibrium". Here is a schematic overview of weighting the tomato and some other experiments:

```
pi| on table |right cup  |     |Half Banana|     |Metal Strip|
ct|AAAAAAAAAA|AAAAA AAAAA|     |AAAAA AAAAA|     |AAAAA AAAAA|
nr|0123456789|01234 56789|     |01234 56789|     |01234 56789|
--|----------------------|     |-----------|     |-----------|
 3|XXXXXXXXXX|           |     |X          |     |X          |
 4| XXXXXXXXX|X          |     |XX         |     |XX         |
 5|X XXXXXXXX| X         |     |X X        |     |X X        |
 6|X  XXXXXXX| XX        |     |X  X       |     |X XX       |
 7|X X XXXXXX| X X       |     |X   X      |><   |X X X      |
 8|X XX XXXXX| X  X      |     |X   X X    |     |X X   X    |
 9|X XX  XXXX| X  X X    |><   |X   X  X   |     |X X   XX   |
10|X XX   XXX| X  X XX   |     |X   X   X  |     |X X   XXX  |><
11|X XX  X XX| X  X X X  |     |X   X    X |><   |X X   XXXX |><
12|X XX  XX X| X  X X  X |><   |X   X     X|><   |X X   XXX X|><
13|X XX  XXX | X  X X   X|><    10001.00000      10100.11100
14|X XX  XXXX| X  X X    |><    Equilibrium      Equilibrium
 9|X XX  XXXX| X  X X    |><    10001.00010      10100.11110
Binary number 01001.10000       10001.00001      10100.11101
```

```
| weight A0 |     |Zero Weight|     |Heavy Metal|
|AAAAA AAAAA|     |AAAAA AAAAA|     |AAAAA AAAAA|
|01234 56789|     |01234 56789|     |01234 56789|
|-----------|     |-----------|     |-----------|
|           |     |X          |     |X          |
| X         |     | X         |     |XX         |
| XX        |     |  X        |     |XXX        |
| XXX       |     |   X       |     |XXXX       |
| XXXX      |     |    X      |     |XXXXX      |
| XXXX X    |     |      X    |     |XXXXX X    |
| XXXX XX   |     |       X   |     |XXXXX XX   |
| XXXX XXX  |     |        X  |     |XXXXX XXX  |
| XXXX XXXX |><   |         X |><   |XXXXX XXXX |
| XXXX XXXXX|><   |          X|><   |XXXXX XXXXX|
 01111.11111       00000.00010       11111.11111
 Theoretical       00000.00001       out of scope
 10000.00000       Theoretical      no equilibrium
                   00000.00000
```

Real Measurement

It is emphasized that our binary balance is in fact: a device for establishing *equality*, in the end, namely between the contents of the right cup and the left cup. In order to be able to proceed, it's important to accept, unconditionally, that such a primitive device is indeed making equality decisions for us. This may be even harder to accept, because the state of equilibrium ($><$) of our balance is not quite an unambiguous concept, as should be obvious by comparison of the two last photographs: (14) and ($15 = 9$), while weighting the tomato in the section *Binary Balance*. Our moderate proposal is therefore to at least agree upon *approximate* equality, denoted as $\approx$.
Other measurements with the same equipment have been done as well: the weight of a Half Banana, the weight of a Metal Strip, the weight of an $A0$ paper weight itself, the weight of nothing (Zero Weight), the weight of too Heavy Metal pieces. With the latter measurements, only the paper weights in the right cup are considered as being informative - because it's evident that all other paper weights are on the table.
A *binary* number can be assigned to the paper weights, as shown: the weights in the right cup correspond with bit 1 and the weights on the table correspond with bit 0. Furthermore, if we adopt the weight of the $A4$ paper as our weighting *unit*, then a binary point must be inserted between the bits for $A4$ and $A5$.
With the above in mind, I think it's best to proceed with the results of the Zero Weight experiment in *Binary Balance*. Replacing equilibrium ($><$) by approximately equality ($\approx$) and thus leaving that decision to the binary balance:

$$00000.00000 \approx 00000.00001 \quad \text{and} \quad 00000.00000 \approx 00000.00010$$

But there is no such agreement upon $00000.00000 \approx 00000.00011$, therefore it must be accepted that:

$$00000.00000 \not\approx 00000.00011$$

Note that all leading zeroes are displayed as well. They tell us how many paper weights there are in the first place. And that most paper weights are on the table. One reason that we start with the Zero Weight is that the the "errors" in zero provides an explanation for the states of equilibrium with the first few other measurements. For example the weight of Half a Banana:

$$10001.00001 = 10001.00000 + 00000.00001$$

$$10001.00010 = 10001.00000 + 00000.00010$$

Or the weight of a Metal Strip ($\pm$):

$$10100.11100 = 10100.11101 - 00000.00001$$

$$10100.11110 = 10100.11101 + 00000.00001$$

Very much the same pattern is recognized with the Tomato measurements: neither of the cups touching the ground - equilibrium / equality - with binary numbers 01001.10000, 01001.10010 and 01001.10001. It's not quite uncommon to write in such a case:

$$\text{tomato weight} = 01001.10001 \pm 00000.00001$$

When written in decimal notation, it reads:

$$0.2^4 + 1.2^3 + 0.2^2 + 0.2^1 + 1.2^0 + 1.2^{-1} + 0.2^{-2} + 0.2^{-3} + 0.2^{-4} + 1.2^{-5}$$

$$\pm 1.2^{-5} = (8 + 1 + 0.5 + 0.03125) \pm 0.03125 = 9.53 \pm 0.03\ A4$$

We were talking about *Real Measurement*, didn't we? When written in grams, the weight of the tomato is:

$$(9.53125 \pm 0.03125).75/16 = 44.67 \pm 0.15\ g$$

What if a weighting experiment is done with the paper weights themselves? The fourth column shows what happens: the $A0$ paper weight can be weighted with all of the other paper weights. This means that the theoretical value 10000.00000 is in equilibrium with a measured value 01111.11111. The reason is that the experimental "erroneous" measurement of $A0$ is obtained, again, by subtracting the (smallest) "error" in Zero Weight from the theoretical $A0$:

$$10000.00000 - 00000.00001 = 01111.11111$$

Quite alike results can be obtained for the other paper weights: 01000.00000 >< 00111.11111, 00100 >< 00011.11111, 00010 >< 00001.11111, 00001.00000 >< 00000.11111, etc. The sequence is ended with the Zero Weight, with other words: nothing in the right cup. Then there are still equilibrium situations, though, with the two lightest weights: $A9$ and $A8$.
Last but not least we have the Heavy Metal weighting experiment, consisting of two times a Metal Strip in the left cup. The experimental outcome, but *without* any error available, is simply: 11111.11111. The theoretical outcome is twice the weight for a single Metal Strip, which indeed is larger than the largest available sum of calibrated weights. In binary:

$$10 \times 10100.11100 = 101001.11000 > 11111.11111$$

The two metal strips are too heavy to be measured with the set of paper weights at hand. Even if we put all available paper weights in the right cup (denoted by 11111.11111) then the left cup still touches the ground.
So far so good for Real Measurement. Results have been made even more trustworthy by switching the (content of the) two cups once in a while. But anyway, we are not going to give any explanation, nor make any excuse, for the errors - systematical or not - in our primitive experimentation. The most important challenge of these sections is: how to deal with bare experimental truth. And simply face the facts, without theorizing too much about them.

Natural Identity

With natural numbers, equality according to Leibniz can indeed be established. Maybe it's a good idea to start again with the simplest representation of the naturals, namely the one that has been described in the section *Keeping a Tally*:

`http://en.wikipedia.org/wiki/Unary_numeral_system`

Properties that are not relevant should be avoided in the first place. With the unary numeral system, for example, it's completely irrelevant whether the tally marks are scribbled with with a pen or whether they are carved in a wooden stick. A property like the following may be considered as relevant, though:

$P_{||||||}(x)$ $:\Longleftrightarrow$ $||||||$ is a substring of the number x in its unary representation

Instead of doing common (ideal) mathematics, it's let's assume that we are in *material mathematics* mode, while nevertheless adopting the common notation for equality:

$$(x = y) \Longleftrightarrow [\ \forall P_k \in I_N : P_k(x) \Longleftrightarrow P_k(y)\ ]$$

According to *Infinitum Actu Non Datur*, there is a *finite* number N of properties in the aspect I_N :

$$P_k(x) :\Longleftrightarrow (k \subset x) \qquad \text{where} \quad k = 1,\ ..\ , N$$

Example. Suppose we have a scope I_7 given by the following properties:

$$I_7 = \{P_1(x), P_2(x), P_3(x), P_4(x), P_5(x), P_6(x), P_7(x)\}$$

Then is $3 = 2$? For the left hand side we have:

$$\begin{array}{lcl}
P_1(3) & :\Longleftrightarrow & | \subset ||| \\
P_2(3) & :\Longleftrightarrow & || \subset ||| \\
P_3(3) & :\Longleftrightarrow & ||| \subset ||| \\
P_4(3) & :\Longleftrightarrow & |||| \not\subset ||| \\
P_5(3) & :\Longleftrightarrow & ||||| \not\subset ||| \\
P_6(3) & :\Longleftrightarrow & |||||| \not\subset ||| \\
P_7(3) & :\Longleftrightarrow & ||||||| \not\subset |||
\end{array}$$

And for the right hand side we have:

$$\begin{array}{lcl}
P_1(2) & :\Longleftrightarrow & | \subset || \\
P_2(2) & :\Longleftrightarrow & || \subset || \\
P_3(2) & :\Longleftrightarrow & ||| \not\subset || \\
P_4(2) & :\Longleftrightarrow & |||| \not\subset || \\
P_5(2) & :\Longleftrightarrow & ||||| \not\subset || \\
P_6(2) & :\Longleftrightarrow & |||||| \not\subset || \\
P_7(2) & :\Longleftrightarrow & ||||||| \not\subset ||
\end{array}$$

Thus we see that $P_3(3) \iff P_3(2)$ is false. Hence the right hand side of the $3 = 2$ identity definition is false. So we conclude that $3 \neq 2$. If making the comparisons $P_k(3) \iff P_k(2)$ is conceived as a sequential process, we can actually take a **Break** after $k = 3$; no further search is needed.
The next example is somewhat less trivial. Assume that we have the same aspect but now ask for equality of the numbers 8 and 9. Then we have for the left hand side:

$$\begin{aligned}
P_1(8) \quad &:\iff \quad | \subset |||||||| \\
P_2(8) \quad &:\iff \quad || \subset |||||||| \\
P_3(8) \quad &:\iff \quad ||| \subset |||||||| \\
P_4(8) \quad &:\iff \quad |||| \subset |||||||| \\
P_5(8) \quad &:\iff \quad ||||| \subset |||||||| \\
P_6(8) \quad &:\iff \quad |||||| \subset |||||||| \\
P_7(8) \quad &:\iff \quad ||||||| \subset ||||||||
\end{aligned}$$

Likewise we have for the right hand side:

$$\begin{aligned}
P_1(9) \quad &:\iff \quad | \subset ||||||||| \\
P_2(9) \quad &:\iff \quad || \subset ||||||||| \\
P_3(9) \quad &:\iff \quad ||| \subset ||||||||| \\
P_4(9) \quad &:\iff \quad |||| \subset ||||||||| \\
P_5(9) \quad &:\iff \quad ||||| \subset ||||||||| \\
P_6(9) \quad &:\iff \quad |||||| \subset ||||||||| \\
P_7(9) \quad &:\iff \quad ||||||| \subset |||||||||
\end{aligned}$$

Sorry, it's because our scope is not wider than this! We conclude that, with respect to our quite limited scope $I_7 : 8 = 9$!
This is utterly absurd! Therefore let us represent each natural as a bit string, as has been done as well in the section *Keeping a Tally.* Such bit strings are quite common, of course, with computer representations. Then another aspect I_N may consist of predicates of the form:

$$P_k(x) \;:\iff \text{ bit } k \text{ of the binary representing the number } x \text{ is up}$$

Certainly the binary representation is much less primitive than the unary one. But even now, according to the basic axiom, the aspect I_N must assumed to be *finite*. In concordance with the *Keeping a Tally* section, we assume that it is six bits wide:

$$I_6 = \{P_0(x), P_1(x), P_2(x), P_3(x), P_4(x), P_5(x)\}$$

Equalities are just trivial as long as they are within the aspect of these six bits. We are not going to repeat these trivialities. But what happens if any (binary) number is beyond that scope? This is clarified by playing the game in *Keeping a Tally* with a number beyond the scope of six bits. It's easy to see that such a game with too large numbers invariably results in a single bit string 111111, indicating *overflow* of the aspect I_6 in use. It's empathized that *any* such too large number results in the *same* bit string. Thus it makes no difference whether we use the unary or whether we use the binary representation:

- all numbers beyond the scope in use are *identical* to each other and identical to the largest number within the same scope

The above is quite in concordance with the fact that integer machine numbers are *exact*, though - due to limitations imposed by hardware - they cannot be infinitely large. If for example the number of binary digits (bits) is 32, then the naturals that can be represented are in the range 0 to $4,294,967,295$. Any number beyond that latter value is beyond the 32-bits scope and thus must be considered as "infinite" - or at least it is not a number (NaN) within the 32-bits aspect. In fact, it's reasonable to say that the number $4,294,967,295$, in this particular sense, *is infinitely large.* To put it in another way: $4,294,967,295$ is a number that is too large. Too large numbers are the *materialization* of certain infinite quantities. And certain infinite quantities are the *idealization* of natural numbers that are too large for unique identification. That's why infinite numbers all "look the same" and, for example, formulas like $\aleph_0 + 1 = \aleph_0$ appear in the idealized domain:

`http://en.wikipedia.org/wiki/Transfinite_number`

Apart from a limited equality, the need for a likewise limited *inequality* shall become apparent soon. It will be found, however, that inequalities like $<$ and $>$, with respect to some aspect I, are a different matter altogether.
So far so good with the naturals, which are representative for the discrete world. Discrete mathematics shall not be the main concern of this thesis anyway. The Mathematical Identity of discrete entities, apart from the idea that they may be too large for identification, does not give rise to any further controversial material.

`https://en.wikipedia.org/wiki/Discrete_mathematics`

Taking the abovementioned internet reference for granted, discrete mathematics doesn't even need any distinction between ideal and material, because idealization can hardly be distinguished from abstraction in this case.

Real Identity

Results of the section *Real Measurement* will be evaluated now. And yes: we shall do some *theory* building. Let's have a closer look at our set of calibrated weights (and calibration as such) in the first place.
There exist several links between *Real Measurement* and the section *Keeping a Tally.* Such as with decimal weights that we did not use, "suitable for general laboratory, commercial, and educational use":

`1kg , 500 g , 2 x 200 g , 100 g , 50 g , 2 x 20 g , 10 g , 5 g ,`
`2 x 2 g, 1 g , 500mg , 2 x 200mg , 100mg , 50mg , 2 x 20mg , 10mg`

In the integer domain of finance, we have an equivalent of this "1-2-2-5" series, "suitable for general commercial use", of course:

Giving rise to all sorts of interesting (and not so interesting) mathematics:

`http://en.wikipedia.org/wiki/Preferred_number`

Weighting with a binary balance is quite analogous to the game played for guessing a number. Replace being a subset ($\subset$) by being smaller ($<$) in the latter and the analogy shall be obvious. Looking at it the other way around,

one could also have said that the weights in the right cup are *part of* ($\subset$) the unknown weight in the left cup of the binary balance.
We claim that our calibration set is the aspect of a Leibniz' Identity for the binary numbers that originate with operating the weighting device. As follows:

$$(x = y) \Longleftrightarrow [\ \forall P_k \in I_N : P_k(x) \Longleftrightarrow P_k(y)\ ]$$

According to *Infinitum Actu Non Datur* there is a *finite* number N of properties in the aspect I_N, the same way as with *Keeping a Tally*:

$$P_k(x)\ \equiv\ \text{bit } k \text{ of the binary representing the number } x \text{ is up}$$

With *Real Measurement* as is, we have a scope I_{10} with ten properties:

$$I_{10} = \{P_0(x), P_1(x), P_2(x), P_3(x), P_4(x), P_5(x), P_6(x), P_7(x), P_8(x), P_9(x)\}$$

The "too heavy" and "too light" decisions (right cup) of our binary balance are responsible for the bit strings. The properties P_k have the following equivalent in experimental physics:

$$P_k = \text{paper weight } Ak \text{ is in the right cup}$$

With our set of calibrated weights, there is **no** equilibrium state (><) for weights greater than 11111.11111 (binary). Such "too large" weights invariably result in the same 11111.11111 bit string. They cannot be distinguished from each other with the equipment at hand. This is quite analogous with the game in *Keeping a Tally* where guessing a too large number (> 63) invariably results in a bit string 111111. Taking for granted that *infinity* is just an idealization of such *too large* numbers, we conclude that there is no essential difference between "infinities" with integer and with real numbers. In computer terms: *overflow* is the same with Natural and with Real Identity.
But how about *underflow*? Of course there are no too small numbers, therefore underflow does not exist, within the realm of the Natural numbers; the smallest number there being 1 and that's it. With *Real Measurement*, there is no such smallest number. The smallest weight in the right cup of a binary balance can be anything. We have stopped cutting pieces of paper in two with the paper weight $A9$. Calibrated paper weights $A10$ and smaller certainly are possible. However, such weights are *too small* to be compared with anything, given the rest of our primitive equipment. It has no sense to extend the aspect of a *Real Identity*, i.e. the calibration set, without improving the *Binary Balance* itself. With the bit of theorizing that has been done, we conclude that, when considering *too small* weights, there appears to be an uncertainty ($\pm$), an *error*, in the (one or two) rightmost bits and this definitely has an influence on all other measurement results. Unlike naturals, real numbers they can be *infinitely small* too, observed as 00000.000?? , with an uncertainty in the (two) rightmost bits. This so-called *error* seems to be present in all (measurable) real numbers less than infinity. It is assumed in the next sections that the unknown weights (left cup) are all within the scope of our calibration set (right cup). Which doesn't mean that we shouldn't remember the *infinitely large* as a special case.

Double Identity

It is possible to have several calibration sets with the same measuring device. And have several measuring devices with the same calibration set, but that's another question. Suppose that we still have the good old set with 10 paper weights, with aspect I_{10}, at our disposal:

$$I_{10} = \{P_0(x), P_1(x), P_2(x), P_3(x), P_4(x), P_5(x), P_6(x), P_7(x), P_8(x), P_9(x)\}$$

The *Zero Weight* experiment shows that our binary balance isn't quite capable to distinguish nothing in the left cup from something in the left cup, namely the two smallest paper weights $A9$, $A8$, perhaps even $A7$. Therefore it may be a sensible decision to remove the three smallest paper weights from our calibration set. Resulting in an even more limited aspect (I_7) of the Leibniz' equality for our binary balance:

$$I_7 = \{P_0(x), P_1(x), P_2(x), P_3(x), P_4(x), P_5(x), P_6(x)\}$$

There is an identity = associated with each of the aspects. Let's distinguish them as follows:

$$(x \stackrel{10}{=} y) \Longleftrightarrow [\ \forall P \in I_{10} : P(x) \Longleftrightarrow P(y)\ ]$$

$$(x \stackrel{7}{=} y) \Longleftrightarrow [\ \forall P \in I_7 : P(x) \Longleftrightarrow P(y)\ ]$$

Unlike with naturals, a finitary identity with real numbers does not prevent *error propagation* with basic arithmetic operations, like for example with (binary) addition. The 10 wide aspect can cause an error with the 7 wide aspect:

$$10100.10\ 110 + 00010.00\ 100 \stackrel{10}{=} 10110.11\ 010$$

However:

$$10100.10 + 00010.00 \stackrel{7}{\neq} 10110.11$$

With other words, a finitary identity, for real numbers anyway, doesn't quite behave like a common equality.
Not only that we have equalities, but, with each of the aspects, there exist *inequalities* as well. Meaningful references are found within the realm of computer technology:

`http://en.wikipedia.org/wiki/Digital_comparator`

"In order to manually determine the greater of two binary numbers, we inspect the relative magnitudes of pairs of significant digits, starting from the most significant bit, gradually proceeding towards lower significant bits until an inequality is found. When an inequality is found, if the corresponding bit of A is 1 and that of B is 0 then we conclude that $A > B$." Example:

```
A = 0100100101010101011010010
B = 0100100101010101010010010
```

This sequential comparison cannot easily be expressed in (parallel) logic, as is the case with Leibniz' *equality*. Neither is the definition applicable to other objects than (integer and real) numbers, expressed as binary strings. Accepting these limitations, the above algorithm can easily be implemented as a (Delphi) Pascal code snippet:

```
program compare;

procedure vgl(A,B : string; bits : integer);
var
  uit : char;
  k : integer;
begin
  if not (Length(A) = Length(B)) then Exit;
  if bits > Length(A)-1 then Exit;
  uit := '=';
  for k := 1 to bits+1 do
  begin
    if (A[k] = '0') and (B[k] = '1') then uit := '<';
    if (A[k] = '1') and (B[k] = '0') then uit := '>';
    if (uit = '<') or (uit = '>') then Break;
  end;
  Writeln(A,' ',uit,' ',B,' (',bits,')');
end;

begin
  vgl('01001.10010','01001.10101',10);
  vgl('01001.10010','01001.10101',7);
  vgl('10000.00000','01111.11111',10);
end.
```

If there is an equality, then the **for**-loop will be completed to the end and essentially nothing happens inside that loop. If there is an inequality, then a **Break** signals that the loop is prematurely ended. This means that *it always takes longer to establish an equality than establish an inequality*. Anyway, the output of the program is:

```
01001.10010 < 01001.10101 (10)
01001.10010 = 01001.10101 (7)
10000.00000 > 01111.11111 (10)
```

It's reasonable to consider the following expressions, with smaller $<$ and greater $>$, as being defined herewith:

$$x \overset{10}{<} y \quad \text{and} \quad x \overset{10}{>} y \qquad \text{and} \qquad x \overset{7}{<} y \quad \text{and} \quad x \overset{7}{>} y$$

Assume quite in general that we have two (binary) aspects, one S (smaller) bits wide, the other T (taller) bits wide and $S \subset T$. The accompanying equalities and inequalities are:

$$\{\overset{S}{<}, \overset{S}{=}, \overset{S}{>}\} \quad \text{and} \quad \{\overset{T}{<}, \overset{T}{=}, \overset{T}{>}\}$$

It's a matter of routine to figure out that the following relationships are valid:

$\overset{S}{<}$	$\overset{T}{<}$		
$\overset{S}{=}$	$\overset{T}{<}$	$\overset{T}{=}$	$\overset{T}{>}$
$\overset{S}{>}$			$\overset{T}{>}$
$\overset{T}{<}$	$\overset{S}{<}$	$\overset{S}{=}$	
$\overset{T}{=}$		$\overset{S}{=}$	
$\overset{T}{>}$		$\overset{S}{=}$	$\overset{S}{>}$

Where it is noted that the lower half of the table can be derived from the upper half. Expanding the shorthands in the table, i.e. being verbose:

- if $x < y$ with respect to the smaller scope
  then $x < y$ with respect to the taller scope
- if $x = y$ with respect to the smaller scope
  then $x < y$ or $x = y$ or $x > y$ with respect to the taller scope
- if $x > y$ with respect to the smaller scope
  then $x > y$ with respect to the taller scope
- if $x < y$ with respect to the taller scope
  then $x < y$ or $x = y$: $x \leq y$ with respect to the smaller scope
- if $x = y$ with respect to the taller scope
  then $x = y$ with respect to the smaller scope
- if $x > y$ with respect to the taller scope
  then $x = y$ or $x > y$: $x \geq y$ with respect to the smaller scope

Loosely speaking, there isn't always a sharp difference: between equal and more or less equal, between less than and less than or equal, between more than and more than or equal.

Real Equality

The fact that an "exact" Leibniz' equality (=) must be distinguished from a "measured" equality (=) in the above sections is not quite satisfactory. What we have in the first place is a Mathematical Identity / Leibniz' equality of real numbers, defined by an aspect based upon calibrated weights. But, more or less apart apart from this, we have an experimental equality as well, which is defined by the equilibrium state of our binary balance. Thus, in order to prevent confusion, it would be handsome to have a different notation for the different equalities. Our proposal is to adopt the following convention:

- Mathematical Identity or exact or Leibniz' equality is written as $\stackrel{C}{=}$
- Experimental Equality or being approximately equal is written as $\approx$

Where it is assumed that mathematical identity is with respect to a calibration set I_C. In order to clarify this issue even further, two possibilities shall be distinguished:

- the calibration set is more accurate than the measuring device
  the measuring device is bottleneck
- the measuring device is more accurate than the calibration set
  the calibration set is bottleneck

What we have seen up to now is the former, but not the latter. Yet the latter possibility is easy to realize with the equipment at hand: it's sufficient to simply remove the three smallest paper weights $A7$, $A8$ and $A9$ from our calibration set. Resulting in an even more limited aspect (I_7) of the Leibniz' equality for our binary balance. We have encountered it already in the *Double Identity* section:

$$I_7 = \{P_0(x), P_1(x), P_2(x), P_3(x), P_4(x), P_5(x), P_6(x)\}$$

There are some dramatic changes in the results of *Real Measurement* if we just do it. And take a good look at *both* cases. Then we find that common equality $(=)$ in mathematics seems to be: *ambiguous*. A far more accurate statement is that the *materialization* of the idealization called equality is actually two-fold:

- with the measuring device as a bottleneck, common equality $(=)$ is observed as an approximate equality $\approx$
- with the calibration set as a bottleneck, common equality $(=)$ is observed as a mathematical identity $\stackrel{C}{=}$

So with any equality $(=)$ of real numbers, we have to distinguish two separate cases: an identity $\stackrel{C}{=}$ or an approximation $\approx$. But apart from these two equalities, we also have to distinguish several *inequalities*. And it shall be obvious that we have to devise distinct notations for them as well, temporary at least. Let's make the following choices, for (non)identity and (non)approximation respectively:

$$\left\{\stackrel{C}{<}, \stackrel{C}{=}, \stackrel{C}{>}\right\} \quad \text{and} \quad \{\prec, \approx, \succ\}$$

The (non)identity is defined in the section *Double Identity*: as equality and inequality with respect to a certain binary scope. The (non)approximation is defined in the section *Real Measurement*: as equilibrium $(\approx)$, left cup with

unknown weight touches the ground ($\prec$). right cup with paper weights touches the ground ($\succ$). The fact that, with the equilibrium state, neither cup touches the ground has been indicated with the symbol $><$, meaning: not greater ($>$), not smaller ($<$), we rather *don't know*: $(\approx) \equiv (\succ\prec)$.
It's a matter of careful observation to figure out that the following relationships are valid:

ID	calibration bottleneck			instrument bottleneck		
$\overset{C}{<}$	$\prec$			$\prec$	$\approx$	
$\overset{C}{=}$	$\prec$	$\approx$	$\succ$		$\approx$	
$\overset{C}{>}$			$\succ$		$\approx$	$\succ$
$\prec$	$\overset{C}{<}$	$\overset{C}{=}$		$\overset{C}{<}$		
$\approx$		$\overset{C}{=}$		$\overset{C}{<}$	$\overset{C}{=}$	$\overset{C}{>}$
$\succ$		$\overset{C}{=}$	$\overset{C}{>}$			$\overset{C}{>}$

Where it is noted that the lower half of the table can be derived from the upper half. Additional observations are:

- with ≈ 0 there can be non-zero paper weights ($A8$, $A9$) in the right cup
  with $\overset{C}{=} 0$ there are no other zeroes than *exactly nothing* in the right cup
- with $\approx$, for the largest paper weight we find $10000.00000 = 01111.11111$
  with $\overset{C}{=}$, for the largest paper weight we find instead $10000.00 \neq 01111.11$
- with $\approx$ there is an *error* in the weight of the tomato and of anything else
  with $\overset{C}{=}$ the weight of the tomato becomes completely *error free* $= 01001.10$
- with $\approx$ there is an equilibrium state ($=$) with every weight in the left cup
  with $\overset{C}{=}$ there is **no** state of equilibrium with certain weights in the left cup
  Consequently with $\overset{C}{=}$, but not with $\approx$, there exist *immeasurable numbers*
- both with $\approx$ and $\overset{C}{=}$ there **is** an equilibrium state ($=$) for too small weights in the left cup. Such too small weights invariably result in a bit string (approximately) zero $= 00000.00(0??)$

With the advent of a bottleneck calibration set (I_7), there actually is a *Double Identity* now for the binary balance. According to the section with the same name, we shall rename the Taller calibration set (I_{10}) to I_T and the Smaller calibration set (I_7) to I_S. Then it's a matter of logic - no observation needed anymore - to see that the above table is translated into the following one:

$\stackrel{S}{<}$	$\stackrel{S}{<}$			$\prec$			$\stackrel{T}{<}$		
$\stackrel{S}{=}$		$\stackrel{S}{=}$		$\prec$	$\approx$	$\succ$	$\stackrel{T}{<}$	$\stackrel{T}{=}$	$\stackrel{T}{>}$
$\stackrel{S}{>}$			$\stackrel{S}{=}$			$\succ$			$\stackrel{T}{>}$
$\prec$	$\stackrel{S}{<}$	$\stackrel{S}{=}$		$\prec$			$\stackrel{T}{<}$		
$\approx$		$\stackrel{S}{=}$			$\approx$		$\stackrel{T}{<}$	$\stackrel{T}{=}$	$\stackrel{T}{>}$
$\succ$		$\stackrel{S}{=}$	$\stackrel{S}{>}$			$\succ$			$\stackrel{T}{>}$
$\stackrel{T}{<}$	$\stackrel{S}{<}$	$\stackrel{S}{=}$		$\prec$	$\approx$		$\stackrel{T}{<}$		
$\stackrel{T}{=}$		$\stackrel{S}{=}$			$\approx$			$\stackrel{T}{=}$	
$\stackrel{T}{>}$		$\stackrel{S}{=}$	$\stackrel{S}{>}$		$\approx$	$\succ$			$\stackrel{T}{>}$

The above may be considered as evidence that the approximate (in)equalities $\{\prec, \approx, \succ\}$, as a figure of speech, are *in between* the Smaller and Taller (non)identities:

$$I_S \subset I_{\prec,\approx,\succ} \subset I_T$$

As a figure of speech, because the problem is that exact (non)identities for approximate (in)equalities do not exist: e.g. $10000.00000 \approx 01111.11111$ while $10000.00000 \stackrel{T}{\neq} 01111.11111$.

Reals in Reality

Real, physical quantities have uncertainties. That is one of the fundamental properties of physics. And it is not just due to quantum considerations. Take an average metal bar. It has no exact length, not even to the precision of an atomic width or so. There are temperature fluctuations and small forces from Brownian motion of the surrounding air which will cause that bar's length to fluctuate. And since the bar itself has a temperature, the atoms themselves are in perpetual motion, which is another cause of inexactness. So there are *fluctuations* of all kind that will cause $x = 0$, when conceived as a physical quantity, to become nonzero. Meaning that $x = 0$ is actually to be *interpreted* as $x = 0 \pm \delta$, where δ is called the uncertainty or the *error*. And if such is the case for 0 , then such is the case for *all real numbers* x , because $x = x + 0$.
But we don't need a physics argument in order to see that there is kind of an uncertainty in *every* realization of the real numbers. Take a look at floating point quantities in a digital computer, especially zero. Assume that our judgement about the standard scope of 32 bits is that it is (too) Small and that a Tall scope with more bits would be better.

```
number 0.00000000000000000000000000000001111011011101110...
scope: |Small----------------------------|Tall-------------> oo
bits :   12345678901234567890123456789012
```

With the *small scope* hardware / software at hand, this value cannot possibly be distinguished from zero. But suppose we buy somewhat better hardware / software with "extended precision" (say 40 bits), such that:

```
number 0.000000000000000000000000000000001111011011101110...
scope: |Small------------------------------------|Tall-----> oo
bits :   1234567890123456789012345678901234567890
```

Then suddenly the value which was supposedly equal to zero becomes *close* to zero instead. It is thus seen that, with digital computers, a real zero can *not really* be distinguished from its (supposedly very small) environment.
More generally, as a consequence of our **Axiom** that *Completed infinity is not given*, we have, for real numbers, that the aspect of their Mathematical Identity can not be infinite, as would be required for most of them (e.g. the irrationals). Therefore, if $x = 0$ with respect a finite aspect S , then it is possible that $x \neq 0$ with respect to a "better" finite aspect T. Hence, for "really real" numbers, it seems that we must take "being zero" with a pinch of salt.
Integer machine numbers are always exact, though not arbitrarily large. Real machine numbers are always approximate (and limited in size as well). The mere fact that integer numbers in a computer are *not* infinitely large has many other consequences, e.g. for carrying out this limit in the non-integer world, the reign of the *real* numbers:

$$\lim_{n \to \infty} 1/n = 0$$

We see, indeed, that the value 0 cannot be reached, by far, for the simple reason that N cannot even be very large on a digital machine. Assuming e.g. standard 32 bits precision for integers, we have:

$$|1/n - 0| < \epsilon \quad \text{whenever} \quad n > N \quad \text{with} \quad \epsilon = 1/N \quad \text{and} \quad N < 2^{32}$$

Real numbers are representative for the continuous world. In *Material* - applied - *Mathematics*, there are limited intervals of the reals, exactly as there are limited intervals of the naturals. Clipping against a viewport is necessary if ideal Euclidean Geometry is meant to be useful with a Computer Graphics application; for the reason that infinitely long straight lines cannot be co-existent with any visual display in the real world. But, with the real numbers, we have an additional complication. Not only that they have a limited range. In applied mathematics, they also have a *limited precision.* According to common mathematics, the real numbers are abundant with *irrationals.* We can even say that the irrationals form the vast majority of the real numbers. But irrationals are defined by a limit (e.g. a Cauchy sequence of some sort). Yet we have equivalents of expressions like $x = \pi$ or $y = \sqrt{2}$ in our programming languages, where it is emphasized that the variables x and y, most of the time, have no more than, say, double precision. This is extremely poor, when compared with exact mathematics. Consequently, numbers like π or $\sqrt{2}$ can by no means be represented "exactly" with such finite precision. If x is the floating point number representing π and δ is the "machine eps" (i.e. an error) then only the following is true:

$$x \not\equiv \pi \quad \text{and} \quad |x - \pi| < \delta$$

Let's continue with an "exact" value of π. Well, not really exact, but far more accurate than with double precision. We invoke Maple for this purpose:

```
> evalf(Pi,30);
                     3.14159265358979323846264338328
```

With a programming language like Delphi (Pascal), the limitations of double precision are clearly shown:

```
program Pie;
begin
  Writeln(Pi:31:29);
end.

D:\jgmdebruijn\Delphi\infinite>Pie
3.14159265358979324000000000000
```

Let's call the Delphi result "small" and the Maple result "tall", then we have the following picture, not with bits but with common decimals:

```
3.14159265358979324000000000000
|-----------------3846264338328------> oo
|small            |tall
```

One may be tempted to think now that the number π thus is equipped with a *Double Identity*, one with the small and one with the tall scope. Then you are almost right, but not quite. The Double Identity would have given rise to *truncation*, but what we see here, and everywhere, is: rounding. *Rounding* has the effect of minimizing the *difference* between the value with the tall scope and the value with the small scope. Instead of:

$$|3.14159265358979323846264338328 - 3.14159265358979323000000000000| =$$

$$0.00000000000000000846264338328$$

we have a smaller outcome for the absolute value of the difference:

$$|3.14159265358979324000000000000 - 3.14159265358979323846264338328| =$$

$$0.00000000000000000153735661672$$

In reality, the mathematical result extends to infinity; it will never end. In order to establish that two arbitrary real numbers are truly *equal*, we thus may have a formidable task, which will possibly never end as well. The situation is radically different, though, for real numbers p and q that happen *not* to be equal:

```
  p = 3.14159265358979323846264338328
  q = 3.14159265358979322846264338328
p-q = 0.00000000000000001000000000000
```

In the above example, it is for sure that $q < p$. And we only need a finite number of comparisons to establish this. In fact, we only need such comparisons for the number zero, because we can always subtract q from p to get $p - q$.
In either case we are lead to the conclusion that what's actually relevant - and feasible - for real numbers is *not the identity* of the two reals but the *approximate equality* of them $x \approx y$. That means: the absolute value of their difference $|x - y| < \delta$ and how close this *error* δ is to zero.

Infinitesimal Equality

A definition of the Mathematical Identity for real numbers has been given. No such definition has been given yet for the Approximate Equality of two reals x and y. But first we have to clarify another issue: what is the precise definition of an *error*? If such is not a *contradictio in terminis*. Well, perhaps it would be a *contradictio in terminis* with Ideal mathematics, but not with Material mathematics, which still is the level of discourse here.
Let there be given two aspects with the above predicates, a smaller one I_S and a taller one I_T (what's in a name), such that I_S is a subset of I_T: $I_S \subset I_T$. Let the size of the smaller aspect be n_S and the size of the taller aspect be n_T. Obviously then $n_S < n_T$. Further details may be filled in with:

`http://en.wikipedia.org/wiki/Calibration`

Making the calibration set a bottleneck must be done in a wise manner, namely such that the left cup or the right cup *barely* touches the ground, if you put the smallest weight (say $A6$) in the restricted calibration set in one of the cups and the other cup is empty. Smaller calibrated weights ($< A6$) should result in equilibrium, or at least a doubtful non-equilibrium. "Ideally, the standard has less than 1/4 of the measurement uncertainty of the device being calibrated." With the primitive equipment at hand, this goal has been fulfilled indeed, though more or less by coincidence. For the binary balance, we have agreed upon $n_T = n_S + 3$ where $n_S = 7$. Quite in general, the following observation can be made now. Let δ be a (presumably small) positive number representing a measurement error, then:

- Any error $\delta \overset{T}{>} 0$ can be made invisible, i.e. $\delta \overset{S}{=} 0$, by narrowing the aspect of the Mathematical Identity, i.e. by making the calibration set a bottleneck, instead of the measuring device.

Note that the above observation is not quite trivial. It is *not* possible to replace an aspect I_{10} by I_7 and suddenly have $10000.00000 \approx 01111.11111$ be changed into $10000.00 \overset{7}{=} 01111.11$. An approximate equality is not a mathematical identity, in general. But *zero is an exception* to the rule.
The following definition of *approximate equality* is proposed. Let δ be the *error* in the measuring instrument (i.e. binary balance device). **Definition:**

$$(x \approx y) \; :\Longleftrightarrow \; \left[\left(x \overset{T}{=} y \right) \vee \left(0 \overset{T}{<} |x - y| \overset{T}{<} \delta \right) \right] \qquad \text{where} \quad \delta \overset{S}{=} 0$$

The fact that approximately zero has a mathematical identity, by exception, has been employed in the definition. Indeed, we do *not* exclude the possibility that x and y are "exactly equal". But, for this reason, it has no sense to assume, once *again*, that $|x - y| = 0$ with respect to I_T. And indeed, if we put x in the left cup and y in the right cup, then it is expected that neither of the cups will touch the ground, because we have agreed on choosing the scope I_S as such.
Example. The weight of the tomato is $W = 01001.10001 \pm 00000.00001$. Hence $W \stackrel{T}{=} 01001.10001$ or $|W - 01001.10001| \stackrel{T}{<} 00000.00111$, if it is assumed, for example, that $\delta \stackrel{T}{=} 00000.00111$. All this is the case with the *full* aspect I_{10}. However, $\delta \stackrel{S}{=} 0$ with a *narrowed* aspect like I_7; the error is "exactly" *zero* then and the weight of the tomato is "exactly" $W \stackrel{S}{=} 01001.10$.
Apart from an approximate equality, approximate *inequalities* shall be defined as well:

$$(x \prec y) \;:\Longleftrightarrow\; \left[(x \not\approx y) \wedge \left(x \stackrel{T}{<} y \right) \right]$$

$$(x \succ y) \;:\Longleftrightarrow\; \left[(x \not\approx y) \wedge \left(x \stackrel{T}{>} y \right) \right]$$

Theorem.

$$x \approx x \quad \text{and} \quad (x \approx y) \Longrightarrow (y \approx x) \quad \text{and} \quad (x \approx y) \wedge (y \approx z) \Longrightarrow (x \approx z)$$

Proof.
$|a - a| \stackrel{T}{<} \delta$
$|a - b| \stackrel{T}{<} \delta \quad \Longrightarrow \quad |b - a| \stackrel{T}{<} \delta$
Hence the reflexive and symmetric properties are trivial. Not quite the same is the case with transitivity:
$|a - b| \stackrel{T}{<} \delta_1$ and $|b - c| \stackrel{T}{<} \delta_2 \quad \Longrightarrow \quad |a - c| \stackrel{T}{\leq} |a - b| + |b - c| \stackrel{T}{<} \delta_1 + \delta_2$
Where $\delta_1 \stackrel{S}{=} 0$ and $\delta_2 \stackrel{S}{=} 0$. How can it be guaranteed that $\delta_1 + \delta_2 \stackrel{S}{=} 0$ as well? The worst case scenario is something like this:

```
delta_1 = 0.00000000000000000000000000111
delta_2 = 0.00000000000000000000000000111
          -------------------------------- +
the sum = 0.00000000000000000000000001110
```

It is thus observed that the aspect necessary for making the sum zero is a bit smaller, literally, than the original one. Thus it is necessary and sufficient to lower the aspect I_S for all of the errors with one bit: $n_S := n_S - 1$. Since the error propagation concerns only two of the kind, it is simply undone herewith. Note. A infinity of transitive equalities would be another matter.
Definition. A real number dx is materialization of an *infinitesimal* - assuming that such an idealization would exist - if the following is valid:

$$0 \stackrel{T}{<} |dx| \stackrel{T}{<} \delta \qquad \text{where} \quad \delta \stackrel{S}{=} 0$$

The number dx may tentatively be called a *finitesimal* at the level of material mathematics. Mind the word *materialization* in the above. Though it seems that the idea of an infinitesimal is conceptually simple, we have the impression that any rigorous treatment, up to date, has been rather cumbersome. Indeed, producing a proper *idealization* of finitesimals is *not* as simple as that, if we believe what's in:

`http://en.wikipedia.org/wiki/Infinitesimal`

So it is tempting to say that, in common mathematics, infinitesimals still are not defined at all. In that case, instead of talking about *finitesimals*, we could claim the more usual (mis)nomer *infinitesimal* for our purpose. Which is precisely what we shall do.
In the material world, due to our double identity, we can have results like this (in binary, where bits between parentheses are with respect to the Tall scope):

$$\sum_{k=0}^{7} 0.00(100) \overset{S}{=} 1$$

We conclude that a sum of zeroes (with respect to S) can indeed be non-zero (with respect to S). The reason is, of course, that there exists another scope and with respect to that taller scope the same infinitesimals are non-zero.
The approximate equality of two real numbers now can be expressed as follows: x and y are approximately equal if they are identical, or if the absolute value of their difference is infinitesimally small. Hence the name *Infinitesimal Equality* for second part of the approximate equality definition and for this section as a whole.