Infinitum Actu Non Datur

[Was nun aber Ihren Beweis für (1) betrifft,] so protestire ich zuvörderst gegen den Gebrauch einer unendlichen Grösse als einer Vollendeten, welcher in der Mathematik niemals erlaubt ist. Das Unendliche ist nur eine façon de parler, indem man eigentlich von Grenzen spricht, denen gewisse Verhältnisse so nahe kommen als man will, während andern ohne Einschränkung zu wachsen verstattet ist. (C.F. Gauss [in a letter to Schumacher, 12 July 1831])
I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction.

`http://en.wikipedia.org/wiki/Carl_Friedrich_Gauss`

Johann Carl Friedrich Gauss (30 April 1777 - 23 February 1855) was a German mathematician and physical scientist who contributed significantly to many fields, including number theory, statistics, analysis, differential geometry, geodesy, geophysics, electrostatics, astronomy and optics. Gauss referred to mathematics as "the queen of *sciences*".

`http://en.wikipedia.org/wiki/Georg_Cantor`

Georg Ferdinand Ludwig Philipp Cantor (March 3 1845 January 6, 1918) was a German mathematician, best known as the inventor of set theory, which has become a fundamental theory in mathematics. Cantor established the importance of one-to-one correspondence between the members of two sets, defined infinite and well-ordered sets, and proved that the real numbers are "more numerous" than the natural numbers. In fact, Cantor's method of proof of this theorem implies the existence of an "infinity of infinities". He defined the cardinal and ordinal numbers and their arithmetic.
In short, Cantor actually defined the *completed infinities*, which, according to Gauss, are never permissible in mathematics:

- **Axiom.** *Completed infinity is not given* (Infinitum Actu Non Datur)

Numerous attempts have been undertaken to *avoid* completed infinities to be incorporated into the foundations of mathematics. Among the more serious attempts, Constructivism and Intuitionism come into mind. But, despite of all the effort spent, common mathematics, with all sorts of completed infinities present in it, is still flourishing. While constructivism and its variants, on the contrary, are still lacking a foundation strong enough to make "Cantorian mathematics" obsolete in for example common calculus. This comprises sort of evidence that such infinitary constructs, in mathematics, can not really be avoided.
We have seen that idealization, hence Ideal Mathematics, is characterized by the involvement of Infinities. It must thus be concluded that Gauss' dictum, our Axiom, cannot be applicable to mathematics at that level. However, in the section *Materialization* we have distinguished two levels of doing mathematics:

- Material Mathematics, as the result of Abstraction/Materialization
- Ideal Mathematics as the result of Idealization of such Abstractions

Therefore, if the Axiom is valid, it can only be applicable to mathematics at the Material, not at the Ideal level. It may thus be decided *not* to adhere to the Axiom for all sort of mathematics, but adopt instead a *weakened version* of it:

- **Axiom.** *Completed infinity is not given* with Material Mathematics

It should be emphasized that the addendum "with Material Mathematics" will not be mentioned explicitly in the sequel at all instances, but shall rather be assumed implicitly. Such is in concordance with the observation that any sharp distinction between ideal and material mathematics is rather artificial.
An apparent reason for restriction of the Axiom to the more "applied" sort of mathematics is the fact that there does not exist a *Physics of Infinity.* It can be asserted with certainty that completed infinities do not belong to empirical science. There are many arguments, but a few of them will serve our purpose for the moment being.
If the universe were infinite in space and time, then it would'nt become dark at night. The whole sky would be as bright as the sun itself. There would be an infinite number of stars, and all their light would have reached us, since there would be an infinite time for it to travel. This phenomenon is known as Olbers' Paradox:

`http://en.wikipedia.org/wiki/Olbers'_paradox`

Fluid flow is described by a set of coupled partial differential equations. It is very clear, though, that this must be considered as a mere illusion. The "differential" volumes are not really infinitely small. Far from that! They should contain a lot of molecules, for the approximation to be valid. There is a beautiful text about this (due to Perrin) in Benoit Mandelbrot's *The Fractal Geometry of Nature*, first few pages (6,7).

As an example how infinities come into "existence" with a common physical theory, here is the well known equation of state for an ideal gas:

$$p.V = n.R.T$$

As soon as the volume V approaches zero (at a given room temperature T) then the pressure p will raise to infinity. Every physicist knows, however, that such is not the case. For a multiple of reasons. The most important being that a real gas does not exactly behave according to the laws for an ideal gas, certainly not for high values of the pressure. Instead of this, it will be subject to a change of state: it will become a fluid in the first place. And it even may become a solid, if pressure continues to squeeze it to still lower values of its volume. If general lessons are to be learned from ideal gas behaviour, it could be something like the following:

- The infinities associated with the ideal gas law are due to idealization of the real gas behaviour. They disappear if the natural gas is modeled more accurately. That is: as soon as mathematical models become more realistic.
- If infinities are likely to occur within the realm of certain physical laws, then the matter subject to these laws will change state, in such a way that infinities are avoided. This effectively means that such "laws" will be no longer valid.

There are some pathological theories in physics where infinities actually do arise, like Quantum Electro Dynamics (QED). In these theories, notions like "point" charges, with infinitely small spatial dimensions, are used explicitly. But what happens there is not a proof that "Actual infinities do exist". Quite on the contrary! We think that innocent looking mathematical concepts like "point" and "continuity" are open to suspicion if they become an integral part of a difficult physical theory. Then the foundations of mathematics suddenly change into a piece of real world physics. Once our mathematical axioms are absorbed by a physical theory, then they become, in fact, assumptions about nature.
Carl Friedrich von Weizsäcker: *.. dass ein Physiker überrasht sein sollte, wenn er Phänomene in der Natur vorfände, in deren Beschreibung das Wort "Unendlich" nicht durch das Wort "sehr gross" ersetzt werden därfte.*

`http://en.wikipedia.org/wiki/Carl_Friedrich_von_Weizs%C3%A4cker`

Acceptance of *Infinitum Actu non Datur* in Applied Mathematics certainly would clarify issues that are related to mathematics in its role of "the queen of sciences", mathematics insofar as it is useful for doing science. Completed infinities in mathematics is one thing, completed infinities in science is quite another matter.

Natural Infinity

According to standard mathematics, *the* Natural Numbers are given. Moreover they are given as a (completed) Infinite Set. This set is commonly denoted as:

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, \ldots\}$$

Theorem. The set of all natural numbers is a completed infinity.
Proof. A set is infinite (i.e. a completed infinity) if there exists a bijection between that set and a proper subset of itself. Now consider the even naturals. They are a proper subset of the naturals and a bijection can be defined between the former and the latter. As follows:

```
2  4  6  8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 ...
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 ...
```

According to the **Axiom** of this thesis, such a Completed Infinity, at some level of mathematics, tentatively called *Material Mathematics*, does not exist. Which isn't going prevent anybody from freely employing certain expressions, like for example $\forall n \in \mathbb{N}$, as a well established ”figure of speech”. So far so good for common mathematics.
How about the mathematics that exists *below* the level of Ideal mathematics? For the sake of clarity, let's take a look at the mathematics laboratory par excellence, the digital computer. How is ”the set of all naturals” represented in there? As numbers running from 1 up to some maximum integer, usually of order 2^{32} or some such. That precise upper bound is not quite relevant. What is relevant is that there indeed **is** a **limited** upper bound. This limited set of naturals is well known in common mathematics too. It is called an **initial segment** of ”the” standard Naturals, denoted as:

$$\mathbb{N}_n = \{1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots, n\}$$

Each such an initial segment contains all the naturals, starting from 1, up to and including a largest natural n. With such an initial segment, it does not follow *per se* that

$$a \in \mathbb{N}_n \quad \text{and} \quad b \in \mathbb{N}_n \quad \Longrightarrow \quad (a + b) \in \mathbb{N}_n$$

Due to the fact that there is a finite upper bound on the initial segment, the sum $(a + b)$ can be beyond that bound. Which is clearly undesirable. Therefore sort of *idealization* is definitely *needed*, namely such that ”beyond a bound” can no longer be the case. A tentative proposal could be to introduce *The* Naturals as the *limit* of an initial segment, where the largest natural n approaches infinity. If you don't like this approach, please note that it will lead, in a few steps, to a result which is actually *standard* mathematics. So you could accept these intermediate steps as sort of heuristic. Whatever, we shall simply assume, for the moment being, that The Naturals *are* defined as:

$$\mathbb{N} = \lim_{n \to \infty} \{1, 2, 3, 4, 5, \ldots, n\}$$

Admittedly, the concept of a limit is somewhat stretched here. What we want to express is that The Naturals are quite resemblant to an initial segment, apart from an upper bound. So why not just say that the naturals *are* an initial segment without any upper bound. A difference with other approaches is that All of The Naturals therefore can *only* be known through the initial segments. Needless to say that Infinity will never be reached Actually - which is typical for limits anyway.
A significant issue in this context is *Cardinality*. When given a completed infinity of naturals, a bijection can set up between all naturals and all even naturals. This is no longer possible when initial segments are the only things that can be known about the naturals. A bijection can still be set up, but it does no longer cover "all" naturals in the segment. Instead, bijection reduces to a process that is known as *counting*. A count E of all even naturals in an initial segment $\mathbb{N}_n = \{1, 2, 3, 4, 5, \ldots, n\}$ where n itself is *even* reveals that $E = n/2$:

```
2  4  6  8 10 12 14 16 18 20 22 24 26 28 30 32 ...  n
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |      |
1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 ... n/2 ...
```

And if n is *odd* in $\mathbb{N}_n$ then $E = (n-1)/2$:

```
2  4  6  8 10 12 14 16 18 20 22 24 26 28 30 ... (n-1)
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |        |
1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 ... (n-1)/2 ...
```

Thus, for n is even, we establish the limit:

$$\lim_{n\to\infty} \frac{\#\text{evens}}{\#\text{all}} = \lim_{n\to\infty} \frac{\#\left(\{1,2,3,4,\ldots,n/2\}\right)}{\#\left(\{1,2,3,4,5,\ldots,n\}\right)} = \lim_{n\to\infty} \frac{n/2}{n} = \frac{1}{2}$$

And, for n is odd, we establish the limit:

$$\lim_{n\to\infty} \frac{\#\text{evens}}{\#\text{all}} = \lim_{n\to\infty} \frac{\#\left(\{1,2,3,\ldots,(n-1)/2\}\right)}{\#\left(\{1,2,3,4,5,6,7,\ldots,n\}\right)} = \lim_{n\to\infty} \frac{(n-1)/2}{n} = \frac{1}{2}$$

With the quotient of the two count functions available, note that it would have been a technicality to define our tentative limit more rigorously. The inevitable conclusion is that the ratio (#evens)/(#all) is equal to 1/2. Therefore the "cardinality" of all even naturals divided by the "cardinality" of all naturals is *not* equal to 1 but equal to 1/2. Mind the scare quotes. Actually the technique as demonstrated is not quite unknown in common mathematics. Look up *Natural Density* and you will find that it's just like it! So it turns out, in the first place, that *no new mathematics* is needed. More interesting facts on this web page:

`http://en.wikipedia.org/wiki/Natural_density`

So here we are! This is what our proposal could have been: simply replace the standard notion of Cardinality by the other *standard* notion of Natural Density. However, experience learns that replacing things by other things is not as simple as that. A better strategy is to leave everything as it is; just consider Natural Densities as more relevant than Cardinalities, if the *finitistic* approach is to be preferred. Thus the gist of the above is not that common mathematics "doesn't know" about "finitistic cardinalities". It does. And the common theory of cardinalities is not so much wrong; I would rather call it *redundant*. Common mathematics may be too much of the good; but maybe that redundancy is inevitable.

PNT look-alikes

One of the most well known theorems in mathematics is the Prime Number Theorem (PNT). The Prime Number Theorem has the same form as the trivial statement that was proved about $E(n)$ = the number of evens, in the previous subsection:

$$\lim_{n\to\infty} \frac{E(n)}{n/2} = 1$$

Let $P(n)$ be the number of primes less than n, where n is a natural number. Then this is the Prime Number Theorem:

$$\lim_{n\to\infty} \frac{P(n)}{n/\ln(n)} = 1$$

Quite another theory says that the cardinality of the prime numbers is *the same* as the cardinality of the natural numbers, which seems contradictory to the asymptotic (natural) density found: $P(n) \approx n/\ln(n)$.
Likewise, the numerosity (cardinality) of the squares equals the numerosity of the naturals. There are "as many" squares as there are naturals. This is known as *Galileo's Paradox*:

`http://en.wikipedia.org/wiki/Galileo%27s_paradox`

Because there exists a bijection between the naturals and the squares:

```
1 2 3  4  5  6  7  8  9  10  11  12  13  ..
| | |  |  |  |  |  |  |   |   |   |   |
1 4 9 16 25 36 49 64 81 100 121 144 169  ..
```

Despite our respect for Galileo's work, let's have another way to look at it:

```
 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 .. 25 .. n
 |     |         |                    |     |
1^2   2^2       3^2                  4^2   5^2
```

Let $Q(n)$ be the number of squares less than or equal to n, where n is a natural. Then we have a PNT look-alike (though much less difficult to prove) theorem:

$$\lim_{n\to\infty} \frac{Q(n)}{\sqrt{n}} = 1$$

Proof. It's easy to see that $\sqrt{n} - 1 < Q(n) \le \sqrt{n}$. Now divide by $\sqrt{n}$ to get $1 - 1/\sqrt{n} < Q(n)/\sqrt{n} \le 1$. For $n \to \infty$ the result follows.
The numerosity (cardinality) of the powers of 7 equals the numerosity of the naturals. There are as many powers of 7 as there are naturals. Proof: there exists a bijection between the naturals and these powers. As depicted here:

```
1  2   3    4     5      6      7       8        9         10  ..
|  |   |    |     |      |      |       |        |          |
7 49 343 2401 16807 117649 823543 5764801 40353607 282475249  ..
```

But let's have another way to look at it:

```
 1    2 3 4 5 6   7    8 9 10 .. 48  49   50 .. 343 .. 2401 ..
 |                |                  |            |       |
7^0              7^1                7^2          7^3     7^4
```

Let $Z(n)$ be the number of 7-powers less than or equal to n, where n is a natural number. Then we have the theorem:

$$\lim_{n\to\infty} \frac{Z(n)}{\ln(n)/\ln(7)} = 1$$

Proof. From a picture [graph of $\ln(x)/\ln(7)$] it's easy to see that:

$$Z(n) \le \frac{\ln(n)}{\ln(7)} < Z(n) + 1 \quad \Longrightarrow \quad \frac{\ln(n)}{\ln(7)} - 1 < Z(n) \le \frac{\ln(n)}{\ln(7)}$$

Divide by $\ln(n)/\ln(7)$ to get:

$$1 - \frac{1}{\ln(n)/\ln(7)} < \frac{Z(n)}{\ln(n)/\ln(7)} \le 1$$

For $n \to \infty$ the result follows easily.
Written as an asymptotic equality: $Z(n) \approx \ln(n)/\ln(7)$. Now suppose that $\aleph_0$ is an incredibly large number - did I suggest anything? - then what if we write:

$$Z(\aleph_0) = \frac{\ln(\aleph_0)}{\ln(7)}$$

Inverse Function Rule

We seek to generalize the above results.

```
   1     2     3     4     5     6    7     8     9    ..
   |     |     |     |     |     |    |     |     |
 A(1)  A(2)  A(3)  A(4)  A(5)  A(6) A(7)  A(8)  A(9)   ..
```

Let there be defined a function $A : N \to N$ on the naturals $\mathbb{N}$. Examples are the squares [$A(n) = n^2$] and the powers of seven [$A(n) = 7^n$] . Assume, in general, that $A(n)$ is a sequence monotonically increasing with n .

The numerosity / cardinality of the function $A(n)$ equals the numerosity of the naturals; there are as many values of A as there are naturals.
Proof: there exists a bijection between the naturals and these values, due to the fact that $A(n)$ is monotonically increasing with n, thus it has no upper bound, as is the case with the naturals too.
But let's have another way to look at it [e.g with $A(n) = 7^n$]:

```
 1   2 3 4 5 6   7  8 9 10 .. 48  49  50 .. 343 .. 2401 ..
 0   0 0 0 0 0   1  1 1  1     1   2   2     3       4   : D(n)
A(0)            A(1)              A(2)      A(3)    A(4)
```

Let $D(n)$ be the number of $A(m)$ values (count) less than or equal to n, where (m, n) are natural numbers. Then we have the following **Theorem**.

$$\lim_{n\to\infty} \frac{D(n)}{A^{-1}(n)} = 1$$

Proof. $A(n)$ is monotonically increasing with n , therefore the inverse sequence $A^{-1}(n)$ exists in the first place. And it is monotonically increasing as well. Furthermore, we see that $D(n)$ is m for $A(m) \le n < A(m + 1)$.
Consequently: $A(D(n)) \le n < A(D(n) + 1)$ hence $D(n) \le A^{-1}(n) < D(n) + 1$ hence $A^{-1}(n) - 1 < D(n) \le A^{-1}(n)$.
Divide by $A^{-1}(n)$ to get: $1 - 1/A^{-1}(n) < D(n)/A^{-1}(n) \le 1$. For $n \to \infty$ now the theorem follows, because $A^{-1}(n) \to \infty$.
My latest information is that this *Inverse Function Rule* - term coined up by Tony Orlow at the internet - goes all the way back to Carl Friedrich Gauss.
Written as an asymptotic equality: $D(n) \approx A^{-1}(n)$. If $\aleph_0$ is a very large number, then we shall write symbolically (and ironically): $Z(\aleph_0) = A^{-1}(\aleph_0)$.
Examples.
$A(n) = n^2 \quad \Longrightarrow \quad \lim_{n\to\infty} D(n)/\sqrt{n} = 1$
$A(n) = 7^n \quad \Longrightarrow \quad \lim_{n\to\infty} D(n)/[\,\ln(n)/\ln(7)\,] = 1$
$A(n) = 2.n \quad \Longrightarrow \quad \lim_{n\to\infty} D(n)/[\,n/2\,] = 1$
Still another example at the Wikipedia page about Fibonacci numbers:

http://en.wikipedia.org/wiki/Fibonacci_number

For large n, the Fibonacci numbers are approximately $F_n \approx \phi^n/\sqrt{5}$. Giving a limit similar to the one for the powers of seven:

$$A(n) = F_n \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{D(n)}{\ln(n.\sqrt{5})/\ln(\phi)} = 1 \qquad \text{where} \quad \phi = \frac{1}{2}(1 + \sqrt{5})$$

Notes.
If the intended use of the Inverse Function Rule would be to find for example a formula for all of the prime numbers, then you would be disappointed. Let it be suggested that $A^{-1}(n) = n/\ln(n)$, but the inverse of this function is useless, because it is not a bijection $\mathbb{N} \to \mathbb{N}$.
Another example. Let $B(n)$ be the number of positive fractions (reducible or

not) with denominators and numerators less than or equal to a natural number n . Then the following is rather trivial - we would rather like to have a formula for *irreducible* fractions instead of this:

$$\lim_{n\to\infty} \frac{B(n)}{n^2} = 1$$

With $A^{-1}(n) = n^2$, therefore $A(n) = \sqrt{n}$, which is also useless because it is not a bijection $\mathbb{N} \to \mathbb{N}$.

Probability Theory

From that internet page about *Natural Densities*:

`http://en.wikipedia.org/wiki/Natural_density`

Quote: *We see that this notion can be understood as a kind of probability of choosing a number*, which obviously is the reason why Natural Densities are *studied in probabilistic number theory*. No big surprise therefore that another application of the above concepts is with *Probability Theory*.
Standard mathematics says that it is *impossible* to have a uniform probability distribution on the naturals. Meaning that it is impossible to have a distribution giving equal probabilities to each of (all) the natural numbers. Not a question, however, about the same for an initial segment of the naturals. The probability for picking, say, the number 7 out of the initial segment $\{1, 2, 3, 4, 5, \ldots, n\}$ is simply: $1/n$. The same holds for an arbitrary number k in that segment. And the sum of all the probabilities is $1/n + 1/n + \ldots + 1/n = n.1/n = 1$, as it should. But something weird happens if we take the limit for $n \to \infty$. Then each of the probabilities for picking one natural number becomes

$$\lim_{n\to\infty} 1/n = 0$$

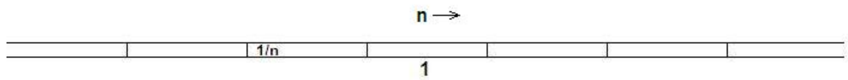
while the sum of all probabilities is still 1, according to

$$\lim_{n\to\infty} 1 = 1$$

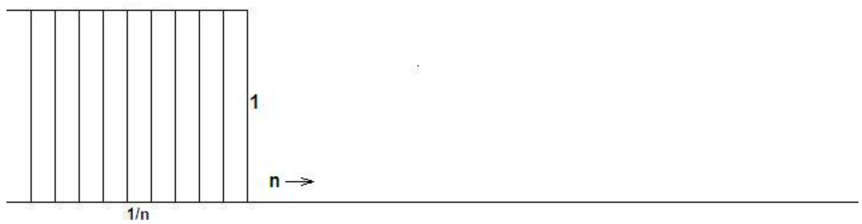
How can this happen? A bunch of zeroes that sums up to one? Oh well, it's not *just* a sum of zeroes. It's an *infinite* sum of zeroes. But nevertheless.
Let's take another example, that only seems remotely resemblant. The following very simple integral is expressed as the limit of a Riemann sum:

$$\int_0^1 dx = \lim_{n\to\infty} \sum_{n=1}^{n} 1/n = \lim_{n\to\infty} n.1/n = \lim_{n\to\infty} 1 = 1$$

Am I blind or what? The only difference with the above "probabilities" is the geometrical interpretation. Here come the probabilities. It is a histogram with height of the blocks $1/n$ and width of the blocks 1 for n blocks. So the total area of the blocks is $(n.1.1/n) = 1$:



And here comes the Riemann sum of the trivial integral. The blocks are $1/n$ wide, 1 high and there are n of them. The total area is $(n.1/n.1) = 1$.



Quite obviously, $\lim_{n\to\infty} n.1/n$ is *exactly the same* algebraic expression with the integral as with our probabilities. But, on the the contrary, it seems that common mathematics has developed quite different ideas about probabilities. The following is a standard **Theorem.**
Let X be a random variable which assumes values in a countable infinite set Q. We can prove there is no uniform distribution on Q .
Proof. Assume there exists such a uniform distribution, that is, there exists $a \geq 0$ such that $P(X = q) = a$ for every $q \in Q$.
Observe that, since Q is countable, by countable additivity of P :

$$1 = P(X \in Q) = \sum_{q\in Q} P(X = q) = \sum_{q\in Q} a$$

Observe that if $a = 0$, $\sum_{q\in Q} a = 0$. Similarly, if $a > 0$, $\sum_{q\in Q} a = \infty$. Contradiction.
Let's analyze. The second part of this reasoning is remotely resemblant to the following *iterated limit*:

$$\lim_{n\to\infty} n \left[\lim_{m\to\infty} \frac{1}{m}\right] = \lim_{n\to\infty} n\ 0 = 0$$

First define uniform probabilities. It's easy to see that these will become zero if the initial segment becomes infinite. Then sum up. Evidently the sum *then* must be zero as well. We have learnt, though, that *iterated limits* may be understood in a quite different way. Due to the following **Theorem**, proved elsewhere by this author:

$$\lim_{x\to a} \left[\lim_{y\to b} F(x,y)\right] = \lim_{(x,y)\to(a,b)} F(x,y)$$

Translated to the present case of interest, with $F(m,n) = n.1/m$:

$$\lim_{n\to\infty} \left[\lim_{m\to\infty} n.\frac{1}{m}\right] = \lim_{(m,n)\to\infty} \left[n.\frac{1}{m}\right]$$

However, the numbers m and n must be *equal*, $m = n$, because they denote one and the same upper bound for the initial segment of the naturals at hand. Therefore we actually have:

$$\lim_{n\to\infty}\left[n.\frac{1}{n}\right] = \lim_{n\to\infty}[1] = 1$$

I think the twist is clear now. But ah, the trouble with understanding all this might be something else as well ..
The sum of infinitely many zeroes in our tentative probability theory is equal to one. The Riemann sum in infinitesimal calculus - mind *infinitesimal* - is equal to one as well. Could it be that the existence of infinitesimals is consequently denied in classical mathematics - because especially in calculus they "aren't needed anymore". But as soon as we add a twist to probability theory, it seems that they turn up again.
As has been said in the beginning, our tentative probability theory is actually equivalent with Natural Densities. The probability that a natural number is e.g. seven is exactly zero, just meaning that the density of seven in the naturals is zero. Yet the density of a natural being an even number is not zero: it equals 1/2. In fact, it has no sense to distinguish Uniform Probabilities on the Naturals from Natural Densities from Finitary Cardinalities. Maybe somewhat exaggerated, but you get the idea. The redundancy, as a consequence of *Infinitum Actu Non Datur* may be summarized loud and clear:

Finitary Cardinalities = Uniform Probabilities = Natural Densities