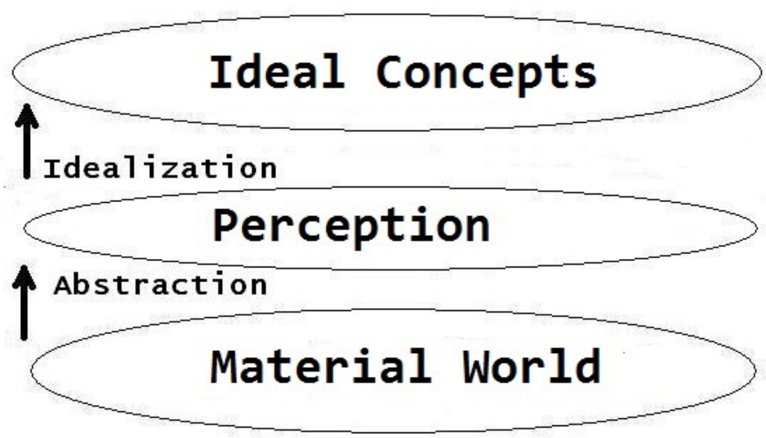
Philosophy of Mathematics

When talking about the *Philosophy of Mathematics*, most of the time, attention is focused upon issues like those mentioned in the book *Gödel, Escher, Bach* by Douglas Hofstadter (ISBN 978 90 2543854 8). To mention a few keywords: formal systems, consistency, completeness, recursion, proposition logic, Turing machines, Church thesis, Artificial Intelligence. It must be said that none of the kind will be considered in this chapter. Instead, we would like to conjecture that two more important mechanisms are involved with a mathematical description of the material world:

- Abstraction, giving rise to Perception
- Idealization, giving rise to Ideal Concepts



Abstraction

Etymology. Perfect passive participle of *abstraho* ("draw away from"). Certain properties of the whole thing are preserved in the process of Abstraction:

http://en.wiktionary.org/wiki/abstract
http://en.wiktionary.org/wiki/abstractus#Latin

We shall argue that Abstraction is *not* a mathematical but rather a *physical* activity. It's already done by our senses. Our eyes can see the light, as it is casted back from a piece of paper. The same piece of paper can be felt by our fingertips. And when it is crumpled up, the sound of it will be heard by our ears. But eyes cannot hear sound, fingertips cannot see light. All these single perceptions of our senses have to work together. And even if we are not handicapped, the end-result is still an abstraction of reality as a whole, a part of it. None of our senses is capable, for example, to see ultraviolet colors, as some insects probably can.
But why should attention be restricted to the creations of Nature? Why not take a look at our own creations: human made Technology? Some cameras are capable to "see" in the infrared domain. Our radio telescopes are even capable to "see" the radio frequencies of far away galaxies. Far more common and well-known everyday abstractions of reality are performed, however, with measuring devices like rods for the abstraction of lengths, clocks for the abstraction of time intervals. But these measuring devices have become more and more self supporting these days. When coupled with digital computers, human interaction is hardly needed anymore. An example is the well known device for the abstraction of weights: a balance. It's further development has resulted in e.g. the fully automated Mettler balance, which is only one of the many examples though. All the apparatus make an abstraction of reality, which is thus a *physical* and not a mental process, let it be that it would be mathematical.
From the Mettler balance example, we see that abstraction results in numbers. Let us not be bothered by the question whether these numbers are numbers in a mathematical sense. Likewise it is assumed that the outcome of abstraction can be another rudimentary mathematical object, like for example a naive "set". Or even a complex number - as far as the latter is concerned, suppose that our measuring device returns an ordered pair or real numbers. And let us not forget geometry. Many laboratory devices come with a screen for displaying graphs. Among the more advanced possibilities offered by modern instruments is three dimensional modeling of the data received. These possibilities only seem to be limited by lack of imagination. It doesn't seem very wise to exclude anything a priori. Therefore we consider any "primitive mathematical notion" as a possible product of abstraction, which is thus a product of *doing*, not of thinking.

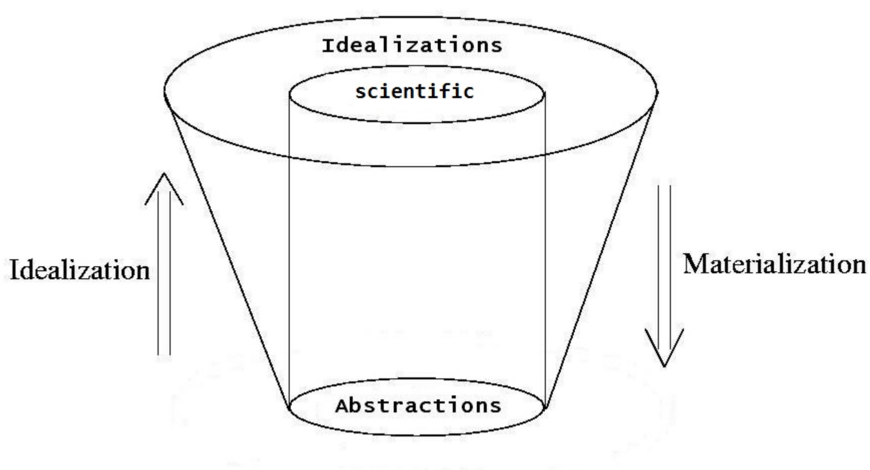
Idealization

This raises an obvious question: where *does* "real" mathematics begin then? Answer: with the next step, **Idealization**.
Yes, idealization could be characterized as the true mathematical activity. But please, tell us what Idealization *is* ! Answer: Idealization is where *imagination and phantasy* come in. It turns out that *infinity* is often a keyword accompanying this process.

Among the most classical examples of idealization, without doubt, is good old Euclidean Geometry - where we should start to consider geometry in its original setting: classical Greek philosophy. Remember utterings like: a point has no size, a line is infinitely thin, parallel lines intersect at infinity. The concept of an irrational number wouldn't have emerged if Euclidean geometry hadn't been there in the first place. Geometry is a playground *par excellence* for infinities.
Yet, strangely enough, most challenging idealizations are not found in pure mathematics, but in theories of Physics. In "The Theory of Heat Radiation" by Max Planck, Wien's Displacement Law (chapter III) can only be derived under the following conditions: if the black radiation contained in a perfectly evacuated cavity with perfectly reflecting walls is compressed or expanded adiabatically and infinitely slowly. Idealized Carnot engines are used in Thermodynamics for defining that stunning but indispensable quantity, called Entropy. And the list goes on and on. How about ideal, frictionless movement in mechanics? How about ideal pendulums, which can only exist through a sine with (almost) zero amplitude. As soon as physicists have devised their mathematical model, then it can be said that idealization has been accomplished a great deal. One should become alerted as soon as the following phrases are being uttered: "perfect", "ideal", "zero", but especially "infinitely", like in "infinitely slow" or "infinitely thin". It can safely be concluded that *Infinities are invariably associated with Idealizations*.

Materialization

Far-fetched as some of them might seem, there is one thing which distinguishes idealizations in Physics from idealizations in Mathematics. It is given by the requirement that there should always be a correspondence between the physical theory and physical experiments. With other words: the process of idealization, in physics, must also exist the other way around. We shall give a name to this inverse process of idealization and call it: **Materialization**.



A little bit of Physics would be NO Idleness in Mathematics. Thus we see that Idealization, as has been conceived by Hilbert and others, is not so much "wrong", but rather too much of the good. It's not enough to have those ideal things in the heaven of Mathematics and be satisfied with it. For the purpose of applications in the real world - that is: *outside* mathematics - it should be possible to revert the whole process, as well. There must be a bijection. Any idealization should be accompanied with a materialization. Besides that Stairway to Heaven, there should exist a Way back to Earth too. For the purpose of applications, mathematical knowledge is supposed to be in concordance with *scientific* standards. How else could it have been so successful in it?
For a mathematical model in Physics, it's relatively easy to establish what a Materialization is all about. All we have to do is to match outcomes of the model with outcomes in the physics laboratory. *Laboratory*, indeed, that could be the keyword! Question is: can we refer to sort of laboratory that is relevant for mathematicians? Yes, we can. With the advent of modern digital computers, mathematicians have become experimentalists too! Every mathematician has a personal laboratory at his disposal these days. And - not surprisingly - with these digital working places, the same kind of limitations are encountered as within the laboratories of physicists. Computers have turned mathematics into a die hard empirical science. Meanwhile, most mathematicians have learned to live with the limitations of the apparatus. They have learned how to cope with roundoff errors, error propagation and the conditioning of matrices. They have learned how to clip straight lines against the finite size of a window. To name only a few of the hurdles that had to be taken in the early days of digital computing. However, physical experiments do not constitute a physical theory. And mathematics cannot be reduced to just the MathLab. There exist at least two basic limitations with any mathematical software experiment:

- in the Discrete domain: the limited size of (the set of) whole numbers
- in the Continuous domain: the limited accuracy of the real numbers

Equivalent with the following MathLab hardware limitations:

- the limited number of words in computer memory
- the limited number of bits in a computer word

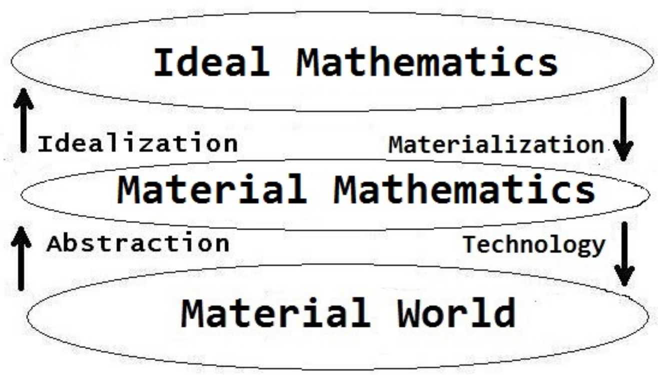
The continuing need for more processing power is closely associated with these lack-of-space experiences. For if you have more space, and more bits, then you need more processing power. And vice versa. Actually, what any mathematician would like to have is an Ideal Computer:

- an unlimited number of words in computer memory
- an unlimited number of bits in a computer word

And all this, of course, associated with unlimited processing power. Look how easy it is to formulate the goals of idealization with nowadays technology in

mind! Nevertheless, these goals have been around for quite a while. Ever since the Ancient Greeks formulated their idealized, Euclidean version of Geometry, to be precise. But this is only part of the story.
In the section *Abstraction* it has been argued that any "primitive mathematical notion" can be a product of abstraction, which we have seen is rather a matter of doing, not so much of thinking. Even such a quite limited form of mathematics nevertheless *is* a form of mathematics. It is suggested herewith that at least two levels of mathematical activity must be distinguished:

- Material Mathematics, as the result of Abstraction/Materialization
- Ideal Mathematics as the result of Idealization of such Abstractions



Computer Graphics and Numerical mathematics may be among Material but not Ideal mathematics. In general, much of *Applied* mathematics should be classified as Material. That's why there exists a *Technology* path from Material Mathematics towards the Material World. Euclidean Geometry and Common Calculus, on the contrary, are typical examples of Ideal - which is virtually the same as "common" - mathematics.
Meanwhile, good old geometry certainly has found its culmination in nowadays computer animation, employed extensively in movies and games: this Material Geometry may be more interesting than Ideal Geometry in some circles. But let us not be distracted by the achievements of Modern Times too quickly.
It is important to observe that there must exist two roads towards Material Mathematics:

- *Bottom up*: by Abstraction from the Material World
- *Top down* : by Materialization from the Ideal World

Last but not least, it should be emphasized that any sharp distinction between Ideal Mathematics and Material Mathematics is somewhat artificial. We shall discover that, in reality, there is often a smooth transition between the two. The extremes may seamlessly fit into each other, in the end.