Falsification of the axiom of infinity

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1 Motivation

The axiom of infinity, when looked at with common sense, is at best absurd: it states that 'some infinite process can be finished'. Even if the axiom is consistent, it leads to theorems that defy any intuition: the Banach-Tarski paradox (for which it is hard to explain why it does not imply that $1 = 2$), the Riemann series theorem (for which it is hard to explain why it does not imply that any number is equal to any number) and the infamous $1+2+3+... = -1/12$. Even if these results are all logically valid and consistent, they do imply there is something wrong with mathematics: they do not, at all, agree with observations in nature. This is not necessarily a problem but it does reduce mathematics to a mere figment of the imagination, and in this reduction the maths loses a lot of its value.

The value of mathematics, besides its beauty, comes from its ability to analytically expand our knowledge in a certain domain. If we note that a certain domain obeys some set of axioms, then we can apply all theorems related to those axioms to that domain. There is no domain, not in physics, not in computer science and not in economics, where the axiom of infinity is valid. No physical system contains an infinite number of particles, no economy has an infinite dollar bill and no computer can construct a set with infinite elements. There are no use-cases for the infinite set outside of mathematics.

As a constructivist I believe all maths is mental: mathematics bears no physical existence, it only exists in the minds of mathematicians. All maths is imaginary and not more real than any other language. The physicist who claims that his formulas are physical nature and therefore bear real existence makes the mistake of thinking that the descriptor and the described are equal. Even though I believe all maths to be imaginary, there are degrees to which a certain mathematical idea is non-existent. Simply put: the infinite set does not fit inside my caveman skull and that is why I consider it be not even an imagination, just an empty term.

2 The proof

I will begin the proof by stating that from which we will derive a contradiction, the axiom of infinity:

$$\exists \omega : (\emptyset \in \omega) \wedge (\forall y : y \in \omega \implies S(y) \in \omega)$$

Where S is defined as $S(x) = x \cup \{x\}$. Some more definitions are required.

Definition 1 (ordinals)**.** An ordinal is either defined as $0 := \emptyset$ or recursively via $n + 1 = S(n)$. The first ordinal which cannot be reached through through the successor operator is ω.

Think of ordinals as just numbers, with the slight catch that their set-nature makes it possible to use "$<$" and "$\in$" completely interchangeably.

Definition 2 (limit supremum of a series of sets)**.** If $(S_n)_{n\in\omega}$ is a series of sets, define the limit as:

$$\overline{\lim_{n\to\omega}} S_n = \bigcap_{m\in\omega} \bigcup_{n\in\omega\setminus m} S_n$$

Where this is the last time you will see "$n \to \omega$" underneath the limit because I will exclusively deal with limits of n going to ω and the overline to emphasize it being the limit supremum will also be omitted, like in the next theorem:

Theorem 1 (Jeltes theorem)**.** *If both* $\lim S_{ni}, \forall i \in I$ *and* $\lim\{S_{ni} : i \in I\} = S$ *exist, then:*

$$\bigcap_{i\in I} \lim S_{ni} \subseteq \bigcup_{S'\in S} S'$$

Proof. First observe that:

$$x \in \bigcup_{S'\in S} S' \iff \exists S' : (x \in S') \wedge (S' \in \bigcap_{m\in\omega} \bigcup_{n\in\omega\setminus m} \{S_{ni} : i \in I\}) \iff$$

$$\exists S' : (x \in S') \wedge (\forall m : \exists n > m : \exists i : S' = S_{ni}) \iff \forall m : \exists n > m : \exists i \in I : x \in S_{ni}$$

$$\iff x \in \bigcap_{m\in\omega} \bigcup_{n\in\omega\setminus m} \bigcup_{i\in I} S_{ni} = \lim \bigcup_{i\in I} S_{ni}$$

So that all that is left is to show that the intersection of the individual limits is inside the union of the global limit. Next we observe that:

$$x \in \bigcap_{i\in I} \lim S_{ni} \iff \forall i \in I : x \in \bigcap_{m\in\omega} \bigcup_{n\in\omega\setminus m} S_{ni} \iff \forall i \in I : \forall m : \exists n > m : x \in S_{ni}$$

But then surely this implies that:

$$\implies \forall m : \exists n > m : \exists i \in I : x \in S_{ni} \iff x \in \bigcap_{m\in\omega} \bigcup_{n\in\omega\setminus m} \bigcup_{i\in I} S_{ni}$$

□

Now all is set for the final theorem, our desired contradiction. We will construct a sequence of sets $\{n+k : k \in \omega\}$ that will violate theorem 1.

Theorem 2 (Inconsistency of infinity)**.** *The existence of ω implies a contradiction.*

Proof. First we claim that:

$$\lim\{n+k : k \in \omega\} = \emptyset$$

Which is the case because:

$$\bigcap_{m\in\omega} \bigcup_{n\in\omega\setminus m} \{n+k : k \in \omega\} = \bigcap_{m\in\omega} \{m+k : k \in \omega\} = \emptyset$$

Next we observe that:

$$\lim n+k = \omega$$

Because:

$$\bigcap_{m\in\omega} \bigcup_{n\in\omega\setminus m} n+k = \bigcap_{m\in\omega} \omega = \omega$$

If we now compare to theorem 1, we will see a contradiction: the inner series is $S_{nk} = n+k$, the outer series is $\{S_{nk} : k \in \omega\}$ and the contradiction is presented as:

$$\bigcap_{k\in\omega} \lim n+k = \bigcap_{k\in\omega} \omega = \omega \nsubseteq \bigcup_{S'\in\lim\{n+k:k\in\omega\}} S' = \bigcup_{S'\in\emptyset} S' = \emptyset$$

□