TripleGrid Calculus

Author: Han de Bruijn
Date: 2006 November

This document is meant as an extension to a previously published one, called *MultiGrid Calculus*. Upon retrospective, *DoubleGrid Calculus* would have been a better name for the latter. The reason being that "Multigrid" is much of a reserved word within the world of Numerical Analysis. Therefore its use in a pure mathematics context will likely give rise to confusion. (But maybe that's just intended?) Anyway, quite unexpectedly, it has been discovered that there exists another kind of Multigrid Calculus, which is distinct from DoubleGrid. It also works with coarsening and refinement of grids, but does not double or halve the intervals. Instead, it makes these intervals larger or smaller, not with a factor two, but with a factor **three**. Ah, and now you could think that the next step is a MultiGrid Calculus employing a factor four or maybe five. But this is not so. The factor four being already covered by a double doubling in the first place. Furthermore, it can be proved that factors five or higher are not an option, except as a powers of 2 and 3 . Thus all possibilities for MultiGrid are exhausted with DoubleGrid and TripleGrid. By the way, the DoubleGrid document has been available all the time at:

`http://hdebruijn.soo.dto.tudelft.nl/hdb_spul/calculus.pdf`

Elimination

This paragraph is essentially a rewrite of a 'sci.math' poster:

`http://groups.google.nl/group/sci.math/msg/f4f1f8e983fef986?hl=en&`

Re: Proof of conjecture wanted.
Consider a piece of an (infinitely) large system of uniform tri-diagonal linear equations:

$$\begin{aligned} -a.T_0 + T_1 - b.T_2 &= 0 \quad (1) \\ -a.T_1 + T_2 - b.T_3 &= 0 \quad (2) \\ -a.T_2 + T_3 - b.T_4 &= 0 \quad (3) \\ -a.T_3 + T_4 - b.T_5 &= 0 \quad (4) \\ -a.T_4 + T_5 - b.T_6 &= 0 \quad (5) \end{aligned}$$

And a boundary condition ($RHS \neq 0$) somewhere, but not within our reach.
It is possiblle to use equations (2) and (4) for eliminating T_2 and T_4 from (3) and replace them by T_1 and T_5. As follows:

$$\begin{aligned} T_2 &= a.T_1 + b.T_3 \quad (2) \\ T_4 &= a.T_3 + b.T_5 \quad (4) \end{aligned}$$

Giving respectively:

$$-a(a.T_1 + b.T_3) + T_3 - b(a.T_3 + b.T_5) = 0 \qquad \Longrightarrow$$

$$-a^2.T_1 + (1-2.a.b)T_3 - b^2.T_5 = 0 \qquad \Longrightarrow$$

$$-\left(\frac{a^2}{1-2ab}\right)T_1 + T_3 - \left(\frac{b^2}{1-2ab}\right)T_5 = 0 \qquad (3,2,4)$$

Thus, by employing this elimination procedure, the matrix pattern:

$$-a \qquad 1 \qquad -b$$

has been replaced by:

$$-\left(\frac{a^2}{1-2ab}\right) \qquad 0 \qquad 1 \qquad 0 \qquad -\left(\frac{b^2}{1-2ab}\right)$$

The above is essentially the theory as has been developed in *MultiGrid Calculus*, the paragraphs *Direct Solver* and *Persistent Properties*.
But we will go even further and eliminate the variables T_1 and T_5 from equation (3) by employing (1) and (6) together with (2) and (4):

$$T_1 = a.T_0 + b.T_2 = a.T_0 + b.(a.T_1 + b.T_3) \qquad (1,2)$$

$$(1-a.b).T_1 = a.T_0 + b^2.T_3 \qquad \Longrightarrow$$

$$T_1 = \left(\frac{a}{1-ab}\right)T_0 + \left(\frac{b^2}{1-ab}\right)T_3$$

$$T_5 = a.T_4 + b.T_6 = a.(a.T_3 + b.T_5) + b.T_6 \qquad (4,5)$$

$$(1-a.b).T_5 = a^2.T_3 + b.T_6 \qquad \Longrightarrow$$

$$T_5 = \left(\frac{a^2}{1-ab}\right)T_3 + \left(\frac{b}{1-ab}\right)T_6$$

Now substitute (1,2) and (4,5) into (3,2,4), then:

$$\begin{aligned} -a^3/(1-2ab)/(1-ab) \quad & T_0 \\ +\left[1-2.a^2.b^2/(1-2ab)/(1-ab)\right] \quad & T_3 \\ -b^3/(1-2ab)/(1-ab) \quad & T_6 \quad = 0 \end{aligned}$$

And the matrix pattern has become:

$$-a^3.X(a,b) \qquad 0 \qquad 0 \qquad 1 \qquad 0 \qquad 0 \qquad -b^3.X(a,b)$$

Where:

$$X(a,b) = \frac{1/(1-2ab)/(1-ab)}{1-2.a^2.b^2/(1-2ab)/(1-ab)} =$$

$$\frac{1}{(1-2ab)(1-ab)-2.a^2.b^2} = \frac{1}{1-3ab}$$

Hence the matrix pattern, with five equations involved, simplified:

$$-\left(\frac{a^3}{1-3ab}\right) \qquad 0 \qquad 0 \qquad 1 \qquad 0 \qquad 0 \qquad -\left(\frac{b^3}{1-3ab}\right)$$

It is seen that the elimination procedure with 5 equations involved results in a pattern which is very much alike the one with 3 equations involved:

$$-\left(\frac{a^2}{1-2ab}\right) \qquad 0 \qquad 1 \qquad 0 \qquad -\left(\frac{b^2}{1-2ab}\right)$$

This would suggest that an elimination procedure with $2n-1$ equations involved would result in a matrix pattern like:

$$-a^n/(1-n.a.b) \quad 0 \quad ... \quad 0 \quad 1 \quad 0 \quad ... \quad 0 \quad -b^n/(1-n.a.b)$$

But this suggestion is false. It is already false for $n=1$, as is clear from $-a \neq -a/(1-ab)$ and $-b \neq -b/(1-ab)$. And it is false for $n=4$, as is clear from $4 = 2\times 2$. Because herewith the function $X(a,b)$ becomes, for $n=4$:

$$X(a,b) = \frac{1/(1-2ab)/(1-2ab)}{1-2\,[a^2/(1-2ab)]\,[b^2/(1-2ab)]}$$

$$\frac{1}{(1-2ab)^2 - 2.a^2.b^2} = \frac{1}{1-4ab+2a^2b^2} \neq \frac{1}{1-4ab}$$

We conclude that the elimination of $(2n-1)$ equations from the given uniform three-diagonal linear system of equations results in a pattern:

$$-a^n/(1-n.a.b) \quad 0 \quad ... \quad 0 \quad 1 \quad 0 \quad ... \quad 0 \quad -b^n/(1-n.a.b)$$

But **if and only if** $(n=2)$ **or** $(n=3)$. The case $(n=2)$ has been treated quite extensively in *MultiGrid Calculus*, which preferrably should have been renamed to *DoubleGrid Calculus*. And the case $(n=3)$ will be treated in the present document, which is properly titled *TripleGrid Calculus*.

Quotient Calculus

So, if we eliminate $(2n-1)$ equations from the uniform three-diagonal system of linear equations, as defined in the preceding paragraph, then we have the following matrix patterns, but **only for** $n=2$ **or** $n=3$:

$$\begin{array}{ccccccc} . & -a^2/(1-2ab) & 0 & 1 & 0 & -b^2/(1-2ab) & . \\ -a^3/(1-3ab) & 0 & 0 & 1 & 0 & 0 & -b^3/(1-3ab) \end{array}$$

So the quotient of the outer diagonal elements, with the variables T_2 and T_4 replaced by T_1 and T_5, for the case $n=2$, is $(b/a)^2$.
And the quotient of the outer diagonal elements, with the variables T_1, T_2, T_4, T_5 eliminated, for the case $n=3$, is $(b/a)^3$.
Thus for $n=2$ and $n=3$ we have that the quotient of the outer diagonals is $(b/a)^n$. After eliminating variables with the $n=2,3$ methods for the first time, all equations can be re-ordered to form two or three blocks in a new but equivalent tridiagonal system. This procedure can be repeated ad infinitum. It should be known from *Multigrid Calculus* that such a thing corresponds with *grid coarsening*, geometrically. Resulting in meshes which are coarsened two- or

threefold every time again. The reverse procedure, corresponding with a two- or threefold mesh refinement, has taking the 2nd or 3rd root from the quotient as its algebraic equivalent. It is known from *Multigrid Calculus* that the $(b/a)^n$ law is even valid for powers $n = 2^m$ where m is a whole number, including zero. And for powers $n = 2^{1/m}$ as well, where m is a whole number except zero. Thus e.g the quotient of the outer diagonal elements for the case $n = 4$ is still as could be conjectured from the cases $n = 2$ and $n = 3$, namely b^4/a^4. In very much the same way, it can be shown that the quotient law is valid for powers $n = 3^m$ and for powers $n = 3^{1/m}$. But, maybe somewhat unexpectedly, it can be proved with my favorite Computer Algebra System, MAPLE, that the conjecture breaks down at $n = 5$:

```
eliminate({-a*T0+T1-b*T2=0,-a*T1+T2-b*T3=0,-a*T2+T3-b*T4=0,
           -a*T3+T4-b*T5=0,-a*T4+T5-b*T6=0,-a*T5+T6-b*T7=0,
           -a*T6+T7-b*T8=0,-a*T7+T8-b*T9=0,-a*T8+T9-b*Ta=0},
           {T1,T2,T3,T4,T6,T7,T8,T9});
```

Giving, among a lot of garbage:

```
      3  3              2  2
{20 b   a  T5 - 21 T5 a   b

      4     4    7     2                    6             5
- 5 b   T5 a  + b   Ta a  + 8 b a T5 - 3 a b   Ta - T5 + b   Ta

      6         5        7     2
- 3 a  b T0 + a   T0 + a   T0 b }]
```

Simplified by hand:

$$
\begin{aligned}
&(20b^3a^3 - 21a^2b^2 - 5b^4a^4 + 8ba - 1)T_5 \\
&+(-3a^6b + a^7b^2 + a^5)T_0 + (-3ab^6 + a^2b^7 + b^5)T_a = 0
\end{aligned}
$$

Which proves that the quotient of the (T_0, T_a) coefficients is **not** equal to $(b/a)^5$. Needless to say that the quotient power law $(b/a)^n$ is also invalid for other numbers not being a power of 2 and/or 3, like for example 7 or 11.
Now imagine successive 2-fold or 3-fold grid coarsenings and refinements. We conclude that these multigrids correspond with an exponent p in $(b/a)^p$ which must be equal to one of the following:

$$p = 3^n/2^m \quad \text{or} \quad p = 2^m/3^n$$

(Because it has no sense to multiply by 3 and then divide by 3 again.) Where m and n are whole numbers. It is not difficult to comprehend that any number p can be approximated to arbitrary accuracy by a suitable choice of m and n .
Theorem. There exist natural numbers m and n such that for a given (positive) real number p and a given uncertainity ϵ the following holds.

$$\left| p - \frac{3^n}{2^m} \right| < \epsilon \quad \text{or} \quad \left| p - \frac{2^m}{3^n} \right| < \epsilon$$

Proof. A well known theorem in number theory is that any irrational number r can be approximated by a fraction m/n, where (m,n) are naturals, such that:

$$\left| r - \frac{m}{n} \right| < \frac{1}{n^2}$$

This theorem can be proven with the Pigeonhole Principle, as exemplified in:

`http://www.cut-the-knot.org/do_you_know/pigeon.shtml`

Or otherwise (with help of the Stern-Brocot tree) as exemplified in:

`http://groups.google.nl/group/sci.math/msg/e62cef86a820d0c3?hl=nl&`

The proof then reads as follows (apart from minor nitpicking). Consider the real number $r = \ln(x.3)/\ln(2)$:

$$\left| \frac{\ln(x.3)}{\ln(2)} - \frac{m}{n} \right| < \frac{1}{n^2}$$

$$-\frac{1}{n^2} < \frac{\ln(x.3)}{\ln(2)} - \frac{m}{n} < \frac{1}{n^2}$$

$$-\frac{\ln(2)}{n} < n\ln(x.3) - m\ln(2) < \frac{\ln(2)}{n}$$

$$2^{-1/n} < \frac{x^n.3^n}{2^m} < 2^{1/n}$$

Now determine x such that $x^n = 1/p$, which is done by $x = \exp(-\ln(p)/n)$:

$$2^{-1/n}p < \frac{3^n}{2^m} < 2^{1/n}p$$

And determine n from a given (relative) uncertainity δ:

$$2^{1/n} = (1+\delta) \quad \Longleftrightarrow \quad n = \frac{\ln(2)}{\ln(1+\delta)}$$

$$2^{-1/n} = \frac{1}{1+\delta} = \frac{(1-\delta)}{1-\delta^2} > 1 - \delta$$

Therefore:

$$(1-\delta)p < 2^{-1/n}p < \frac{3^n}{2^m} < 2^{1/n}p = (1+\delta)p$$

$$|p - \frac{3^n}{2^m}| < \delta p$$

At last, eventually replace the relative error by an absolute error, i.e. $\delta p = \epsilon$, and we are done. By the way, then:

$$n = \frac{\ln(2)}{\ln(1+\epsilon/p)} \approx \frac{\ln(2)}{\epsilon/p} \quad \Longrightarrow \quad n \approx \frac{p}{\epsilon} \qquad \text{(order of magnitude)}$$

Thus an estimate for the powers in $3^n/2^m$ is one divided by the allowed relative error in the approximation.
The above is the completion of a missing link in *Multigrid Calculus*. Up to now, I could only prove that p should be of the form 2^n or $1/2^n$ with no hope of extending this to arbitrary real exponents. The problem is stated on page 15 of the paper where it says "We seek to generalize" and then followed by an unsatisfactory solution.

DoubleGrid Product

With well known formulas for the trigonometric and hyperbolic functions, the following results can be derived.
Trigonometric case:

$$\begin{aligned} & 2+2\cos(2\phi) = 2+2\left[2\cos^2(\phi)-1\right] = 4\cos^2(\phi) \\ \Longrightarrow \quad & 2+2\cos(2\phi) = \left[\{2+2\cos(\phi)\}-2\right]^2 \end{aligned}$$

Hyperbolic case:

$$\begin{aligned} & 2+2\cosh(2p) = 2+2\left[2\cosh^2(p)-1\right] = 4\cosh^2(p) \\ \Longrightarrow \quad & 2+2\cosh(2p) = \left[\{2+2\cosh(p)\}-2\right]^2 \end{aligned}$$

Summarizing both cases:

$$\begin{aligned} y(\phi) = 2+2\cos(\phi) \quad & \Longrightarrow \quad y(2\phi) = [\,y(\phi)-2\,]^2 \\ y(p) = 2+2\cosh(p) \quad & \Longrightarrow \quad y(2p) = [\,y(p)-2\,]^2 \end{aligned}$$

The product of the outer diagonal elements with DoubleGrid Calculus has been presented in the paragraph *Product Function* of the corresponding document:

$$a'.b' = \frac{a^2}{1-2.a.b}\;\frac{b^2}{1-2.a.b} = \left(\frac{a.b}{1-2.a.b}\right)^2$$

Thus the grid doubling iterations are of the form:

$$x := \left(\frac{x}{1-2x}\right)^2 = \left(\frac{1}{1/x-2}\right)^2$$

The latter meaning that we could consider instead the iterations:

$$1/x := (1/x-2)^2 \quad \text{or} \quad y := (y-2)^2$$

Now remember the result we have just derived:

$$\begin{aligned} y(\phi) = 2+2\cos(\phi) \quad & \Longrightarrow \quad y(2\phi) = [\,y(\phi)-2\,]^2 \\ y(p) = 2+2\cosh(p) \quad & \Longrightarrow \quad y(2p) = [\,y(p)-2\,]^2 \end{aligned}$$

And it is apparent that these formulas both seem to cover the iterations. The trigonometric formula is valid for $0 \le y \le 4$. The hyperbolic formula is valid for $y \ge 4$. Nothing new, actually. This is entirely equivalent with statements about DoubleGrid in *The Trigonometric Connection* and *The Hyperbolic Connection* within the corresponding document.
It's a matter of routine now to prove that there is a set of closed formulas for the DoubleGrid iterands. Start with $y = y(\phi)$ or $y = y(p)$ as the zero'th iterand. Then we have $y(2\phi)$ or $y(2p)$ as the first iterand, $y(4\phi)$ or $y(4p)$ as the second iterand, and so on. In general: $y(2^n\phi)$ or $y(2^n p)$ as the n-th iterand. Working back to the original variables, we have for the product of the outer-diagonal elements, after n grid doublings:

$$x_n = \begin{cases} 1/\left[\,2 + 2\cosh(2^n p)\,\right] & \text{for} \quad 0 < x_n \le 1/4 \\ 1/\left[\,2 + 2\cos(2^n \phi)\,\right] & \text{for} \quad x_n \ge 1/4 \end{cases}$$

The essentials are also found in an old 'sci.math' poster:

`http://groups.google.nl/group/sci.math/msg/ca645eeb6d038b1f?hl=en&`

Re: Induction with a hard start
It is established in *MultiGrid Calculus* that values x_n cannot "escape" from their intervals. Thus either all iterands are $0 < x_n \le 1/4$ or all iterands are $x_n \ge 1/4$. And there is no way out. Furthermore, the formulas indicate that it's more handsome to start with any (hyperbolic) angle and work from there - simply by doubling the (hyperbolic) angles - rather than trying to determine an initial (hyperbolic) angle p or ϕ from: $\cosh(p) = 1/(2.x) - 1$ or $\cos(\phi) = 1/(2.x) - 1$.

TripleGrid Product

With well known formulas for the trigonometric and hyperbolic functions, the following results can be derived.
Trigonometric case:

$$\begin{aligned} & 2 + 2\cos(3\phi) = 2 + 2\left[\cos(2\phi)\cos(\phi) - \sin(2\phi)\sin(\phi)\right] = \\ & 2 + 2\left[\left\{2\cos^2(\phi) - 1\right\}\cos(\phi) - \left\{2\sin(\phi)\cos(\phi)\right\}\sin(\phi)\right] = \\ & 2 + 2\left[2\cos^3(\phi) - \cos(\phi) - 2\left\{1 - \cos^2(\phi)\right\}\cos(\phi)\right] = \\ & 2 + 2\left[2\cos^3(\phi) - \cos(\phi) - 2\cos(\phi) + 2\cos^3(\phi)\right] = \\ & 2 + 2\left[4\cos^3(\phi) - 3\cos(\phi)\right] = 8\cos^3(\phi) - 6\cos(\phi) + 2 \\ \Longrightarrow \quad & 2 + 2\cos(3\phi) = 8\cos^3(\phi) - 6\cos(\phi) + 2 \end{aligned}$$

Try to divide the outcome by $[2 + 2\cos(\phi)]$:

$$\begin{aligned} & \frac{8\cos^3(\phi) - 6\cos(\phi) + 2}{2\cos(\phi) + 2} = 4\cos^2(\phi) - 4\cos(\phi) + 1 = \\ & [2\cos(\phi) - 1]^2 \\ \Longrightarrow \quad & 2 + 2\cos(3\phi) = [2 + 2\cos(\phi)]\left[\{2\cos(\phi) + 2\} - 3\right]^2 \end{aligned}$$

Hyperbolic case:

$$\begin{aligned}
& 2+2\cosh(3p) = 2+2\left[\cosh(2p)\cosh(p)+\sinh(2p)\sinh(p)\right] = \\
& 2+2\left[\left\{2\cosh^2(p)-1\right\}\cosh(p)+\left\{2\sinh(p)\cosh(p)\right\}\sinh(p)\right] = \\
& 2+2\left[2\cosh^3(p)-\cosh(p)+2\left\{\cosh^2(p)-1\right\}\cosh(p)\right] = \\
& 2+2\left[2\cosh^3(p)-\cosh(p)-2\cosh(p)+2\cosh^3(p)\right] = \\
& 2+2\left[4\cosh^3(p)-3\cosh(p)\right] = 8\cosh^3(p)-6\cosh(p)+2 \\
\Longrightarrow\quad & 2+2\cosh(3p) = 8\cosh^3(p)-6\cosh(p)+2
\end{aligned}$$

Try to divide the outcome by $[2+2\cosh(p)]$:

$$\begin{aligned}
& \frac{8\cosh^3(p)-6\cosh(p)+2}{2\cosh(p)+2} = 4\cosh^2(p)-4\cosh(p)+1 = \\
& [2\cosh(p)-1]^2 \\
\Longrightarrow\quad & 2+2\cosh(3p) = [2+2\cosh(p)]\left[\{2\cosh(p)+2\}-3\right]^2
\end{aligned}$$

Summarizing both cases:

$$\begin{aligned}
y(\phi) = 2+2\cos(\phi) \quad &\Longrightarrow \quad y(3\phi) = y(\phi)\left[\,y(\phi)-3\,\right]^2 \\
y(p) = 2+2\cosh(p) \quad &\Longrightarrow \quad y(3p) = y(p)\left[\,y(p)-3\,\right]^2
\end{aligned}$$

The product of the outer diagonal elements with TripleGrid Calculus is derived easily with help of the *Elimination* paragraph:

$$a'.b' = \frac{a^3}{1-3.a.b}\,\frac{b^3}{1-3.a.b} = a.b\left(\frac{a.b}{1-3.a.b}\right)^2$$

Thus the grid tripling iterations are of the form:

$$x := x\left(\frac{x}{1-3x}\right)^2 = x\left(\frac{1}{1/x-3}\right)^2$$

The latter meaning that we could consider instead the iterations:

$$1/x := 1/x(1/x-3)^2 \quad \text{or} \quad y := y(y-3)^2$$

Now remember the result we have just derived:

$$\begin{aligned}
y(\phi) = 2+2\cos(\phi) \quad &\Longrightarrow \quad y(3\phi) = y(\phi)\left[\,y(\phi)-3\,\right]^2 \\
y(p) = 2+2\cosh(p) \quad &\Longrightarrow \quad y(3p) = y(p)\left[\,y(p)-3\,\right]^2
\end{aligned}$$

And it is apparent again that these formulas both seem to cover the iterations. The trigonometric formula is valid for $0 \le y \le 4$. The hyperbolic formula is valid for $y \ge 4$. This **must** be the case, because of some results obtained in the DoubleGrid document, the paragraph *Governing Equation*. Here it is established that the *dangerous* and the *safe* domains of Gaussian Elimination have

a meaning which is independent of grid coarsenings and refinements. It is entirely determined by the nature (i.e. discriminant) of a second order differential equation with constant coefficients, which is accompanying our uniform three-diagonal linear system of equations. It remains somewhat surprising, though, that the the domains $0 < y \le 4$ and $y \ge 4$ are independently and seamlessly recovered with TripleGrid as well.
It's a matter of routine now to prove that there is a set of closed formulas for the TripleGrid iterands. Start with $y = y(\phi)$ or $y = y(p)$ as the zero'th iterand. Then we have $y(3\phi)$ or $y(3p)$ as the first iterand, $y(9\phi)$ or $y(9p)$ as the second iterand, and so on. In general: $y(3^n\phi)$ or $y(3^n p)$ as the n-th iterand. Working back to the original variables, we have for the product of the outer-diagonal elements, after n grid triplings:

$$x_n = \begin{cases} 1/\left[\,2 + 2\cosh(3^n p)\,\right] & \text{for} \quad 0 < x_n \le 1/4 \\ 1/\left[\,2 + 2\cos(3^n \phi)\,\right] & \text{for} \quad x_n \ge 1/4 \end{cases}$$

Again, it's more handsome to start with any (hyperbolic) angle and work from there - simply by tripling the angles - rather than trying to determine an initial angle p or ϕ from: $\cosh(p) = 1/(2.x) - 1$ or $\cos(\phi) = 1/(2.x) - 1$.

Stationary Points

The k-th iterate of one divided by the product of the outer diagonal elements of our - meanwhile quite quite familiar - uniform three-diagonal matrix, is given by:

$$y_k = 2 + 2\cos(\phi_k) \quad \text{or} \quad y_k = 2 + 2\cosh(p_k)$$

Depending on whether y is in the *dangerous domain* $0 \le y \le 4$ or in the *safe domain* $y \ge 4$ respectively. With *DoubleGrid* coarsening, the successive angles are given by doubling:

$$\phi_{k+1} = 2\phi_k \quad \text{or} \quad p_{k+1} = 2p_k$$

Which is equivalent to the law:

$$y_{k+1} = (y_k - 2)^2$$

With *TripleGrid* coarsening, the successive angles are given by tripling:

$$\phi_{k+1} = 3\phi_k \quad \text{or} \quad p_{k+1} = 3p_k$$

Which is equivalent to the law:

$$y_{k+1} = y_k(y_k - 3)^2$$

Stationary points are easily detected with the method of the angles. This has been discussed at length for *DoubleGrid* in the accompanying document. So let's give here a few TripleGrid examples. Starting with:

$$y_k = y_k(y_k - 3)^2 \quad \Longleftrightarrow \quad \left[(y_k - 3)^2 - 1\right] y_k = 0 \quad \Longleftrightarrow$$

$$y_k(y_k^2 - 6y_k + 8) = 0 \quad \Longleftrightarrow \quad y_k(y_k - 2)(y_k - 4) = 0$$

Solutions:

$$y_k = \{ \ 0 \ , \ 2 \ , \ 4 \ \}$$

Or equivalently, and perhaps easier:

$$\cos(3\phi_k) = \cos(\phi_k) \quad \Longleftrightarrow \quad 3\phi_k = \{ \ \phi_k \ , \ -\phi_k + 2\pi \ , \ \phi_k + 2\pi \ \} \quad \Longleftrightarrow$$

$$\phi_k = \{0 \ , \ \pi/2 \ , \ \pi \ \} \quad \Longleftrightarrow \quad y_k = 2 + 2\cos(\phi_k) = \{4 \ , \ 2 \ , \ 0 \ \}$$

However, these equations may give rise to new questions. We may ask, namely, what *all* solutions of the following equations are:

$$\begin{aligned} y_k(y_k - 3)^2 &= 4 \\ y_k(y_k - 3)^2 &= 2 \\ y_k(y_k - 3)^2 &= 0 \end{aligned}$$

The first equation is equivalent with:

$$y_k^3 - 6y_k^2 + 9y_k - 4 = 0$$

We know that $y_k = 4$ is a root. Thus another dividing polynomial can be found with a long division:

$$\frac{y_k^3 - 6y_k^2 + 9y_k - 4}{y_k - 4} = y_k^2 - 2y_k + 1 = (y_k - 1)^2$$

The second equation is equivalent with:

$$y_k^3 - 6y_k^2 + 9y_k - 2 = 0$$

We know that $y_k = 2$ is a root. Thus another dividing polynomial can be found with a long division:

$$\frac{y_k^3 - 6y_k^2 + 9y_k - 2}{y_k - 2} = y_k^2 - 4y_k + 1 = \left[y_k - (2 - \sqrt{3})\right] \left[y_k - (2 + \sqrt{3})\right]$$

The third equation is trivial and has $y_k = 3$ as an extra solution. In total we find with triple fold grid coarsening the following stationary values:

$$\begin{array}{rcccc} 1 & \Longrightarrow & 4 & \Longrightarrow & 4 \\ 2 - \sqrt{3} & \Longrightarrow & 2 & \Longrightarrow & 2 \\ 2 + \sqrt{3} & \Longrightarrow & 2 & \Longrightarrow & 2 \\ 3 & \Longrightarrow & 0 & \Longrightarrow & 0 \end{array}$$

Another example, as has been formulated and solved with my favorite Computer Algebra System (MAPLE):

```
> y1 := series(y0*(y0-3)^2,y0);
                                          2     3
                      y1 := 9 y0 - 6 y0  + y0

> y2 := series(y1*(y1-3)^2,y0,10);

                   2           3           4          5          6
  y2 := 81 y0 - 540 y0  + 1386 y0  - 1782 y0  + 1287 y0  - 546 y0  +

              7        8     9
        135 y0  - 18 y0  + y0

> solve(y0 = series(y1*(y1-3)^2,y0,10),y0);

                  1/2          1/2                                 1/2          1/2
                 5            5           1/2        1/2          5            5
  0, 2, 4, 3/2 + ----, 3/2 - ----, 2 + 2   , 2 - 2   , 5/2 + ----, 5/2 - ----
                  2            2                                  2            2
```

Impressive as it may seem, the method of angles enables us to solve this whole thing entirely by hand:

$$\cos(9\phi_k) = \cos(\phi_k) \quad \Longleftrightarrow \quad 9\,\phi_k = \pm\phi_k + n.2\pi \quad \text{where} \quad n = 0\,,\,1\,,\,2\,,\,3\,,\;4$$

$$\phi_k = \{\,0\,,\;\pi/4\,,\;\pi/2\,,\;3\pi/4\,,\;\pi\,,\;\pi/5\,,\;2\pi/5\,,\;3\pi/5\;\,,\;4\pi/5\,\}$$

The angles with $\pi/5$ in them can be found with help of the following document:

`http://hdebruijn.soo.dto.tudelft.nl/hdb_spul/cospower.pdf`

Where a useful formula, for our purpose, is found:

$$\cos(5x) = 5\cos(x) - 20\cos^3(x) + 16\cos^5(x)$$

Substitute herein $x = \pi/10$, then we have, with $z = \cos(\pi/10)$:

$$0 = 5z - 20z^3 + 16z^5 \quad \Longleftrightarrow \quad 16(z^2)^2 - 20(z^2) + 5 = 0 \quad \Longrightarrow$$

$$z^2 = \frac{20 \pm \sqrt{20^2 - 4.5.16}}{32} = \frac{5 \pm \sqrt{5}}{8}$$

Because it is certain that $z \neq 0$. Furthermore it is evident that we should take the largest root. Thus:

$$\cos(\pi/5) = 2\cos^2(\pi/10) - 1 = \frac{5+\sqrt{5}}{4} - 1 \quad = \quad \frac{+1+\sqrt{5}}{4}$$
$$\cos(2\pi/5) = 2\cos^2(\pi/5) - 1 = 2\frac{1+5+2\sqrt{5}}{16} - 1 \quad = \quad \frac{-1+\sqrt{5}}{4}$$
$$\cos(3\pi/5) = \cos(\pi - 2\pi/5) = -\cos(2\pi/5) \quad = \quad \frac{+1-\sqrt{5}}{4}$$
$$\cos(4\pi/5) = \cos(\pi - \pi/5) = -\cos(\pi/5) \quad = \quad \frac{-1-\sqrt{5}}{4}$$

Herewith we finally find:

$$y_k = 2 + 2\cos(\phi_k) =$$

$$\left\{ 4\ ,\ 2+\sqrt{2}\ ,\ 2\ ,\ 2-\sqrt{2}\ ,\ 0\ ,\ \frac{5+\sqrt{5}}{2}\ ,\ \frac{3+\sqrt{5}}{2}\ ,\ \frac{5-\sqrt{5}}{2}\ ,\ \frac{3-\sqrt{5}}{2} \right\}$$

Which is, of course, an exact match with the MAPLE solution.
At last it can be tracked how the stationary points, as found here, transform into each other, as calculated from the formula $y_k(y_k-3)^2 \Longrightarrow y_{k+1}$:

$$\begin{array}{ccccc} 2+\sqrt{2} & \Longrightarrow & 2-\sqrt{2} & \Longrightarrow & 2+\sqrt{2} \\ \frac{5+\sqrt{5}}{2} & \Longrightarrow & \frac{5-\sqrt{5}}{2} & \Longrightarrow & \frac{5+\sqrt{5}}{2} \\ \frac{3+\sqrt{5}}{2} & \Longrightarrow & \frac{3-\sqrt{5}}{2} & \Longrightarrow & \frac{3+\sqrt{5}}{2} \end{array}$$

Grid refinement

So far so good for grid *coarsening*. Let's consider instead the inverse of grid coarsening, which is *grid refinement*. Now it is quite clear that - with DoubleGrid as well as with TripleGrid - all iterations are exactly the reverse of the above. Thus, for DoubleGrid refinement:

$$y_k = (y_{k+1} - 2)^2$$

This actually means that we have to solve a quadratic equation for y_{k+1} :

$$y_{k+1}^2 - 4y_{k+1} + 4 - y_k = 0$$

The discriminant D of a quadratic equation $a.x^2+b.x+c = 0$ is very well known:

$$D = b^2 - 4.a.c$$

Hence, for the problem at hand, the discriminant is:

$$D = 16 - 4(4 - y_k) = 4y_k$$

And it is positive for $y_k \geq 0$. The solutions of the quadratic equation are then:

$$y = 2 \pm \sqrt{y_k}$$

But these solutions may *both* serve again as input for a next iteration, associated with DoubleGrid mesh refinement. This means that $\sqrt{y_k} \leq 2$ and therefore $y_k \leq 4$. Thus we have restricted the possible values of y_k to the interval $[0, 4]$. (The inverse of) this interval is known in MultiGrid (read: DoubleGrid) Calculus as the *Dangerous Interval*. The whole danger being a division by zero in the accompanying Gaussian elimination process. We can see that the restriction $0 \leq y_k \leq 4$ is both necessary and sufficient. Because it is impossible for the

iterands y_k to "escape" from that interval.
If, on the other hand $y_k > 4$, then one of the roots becomes negative, while the other remains positive and becomes greater than 4 again. With the next iteration, only the positive root can be used, resulting again in a negative and a positive root. The latter becomes larger and larger. This form of DoubleGrid refinement has been advertized as the *Safe Interval* in *Multigrid Calculus*.
But, as before, it should have been easier to employ the angles instead:

$$\cos(\phi_k) = \cos(2\phi_{k+1}) \quad \Longleftrightarrow \quad \phi_{k+1} = \left\{ \frac{\phi_k}{2} \, , \, \pi - \frac{\phi_k}{2} \right\}$$

Now we will start with the value that is certainly *dangerous*, namely $y_0 = 0$. Because that value will give rise to a division by zero during the Gaussian elimination process. Then we have:

$$\cos(\phi_0) = \frac{1}{2}y_0 - 1 = -1 \quad \Longrightarrow \quad \phi_0 = \pi$$

The next step is:

$$\phi_1 = \left\{ \frac{\pi}{2} \, , \, \pi - \frac{\pi}{2} \right\} = \pi \left\{ \frac{1}{2} \right\}$$

Add to this the initial value π , then we have in total for the dangerous values so far:

$$\phi_1 \cup \phi_0 = \pi \left\{ \frac{1}{2} \, , \, 1 \right\}$$

The next step is:

$$\phi_2 = \left\{ \frac{\pi}{4} \, , \, \pi - \frac{\pi}{4} \right\} = \pi \left\{ \frac{1}{4} \, , \, \frac{3}{4} \right\}$$

Add to this the other iterands, then we have in total for the dangerous values so far:

$$\phi_2 \cup \phi_1 \cup \phi_0 = \pi \left\{ \frac{1}{4} \, , \, \frac{1}{2} \, , \, \frac{3}{4} \, , \, 1 \right\} = \pi \left\{ \frac{1}{4} \, , \, \frac{2}{4} \, , \, \frac{3}{4} \, , \, \frac{4}{4} \right\}$$

Now it's not difficult anymore to make an educated guess for the next steps. And it is conjectured that:

$$\phi_n \cup \phi_{n-1} \cup ... \cup \phi_2 \cup \phi_1 \cup \phi_0 = \frac{\pi}{2^n} \left\{ 1 \, , \, 2 \, , \, 3 \, , \, \ldots , 2^n \right\}$$

The physical meaning of this is that (the angles of the) dangerous points are arbitrarily *dense* in the domain $0 < y_n \le 4$. Sooner or later, any number in the dangerous domain will become a candidate for division by zero in the (iterative version of the) Gaussian elimination process. But the above is only a replay of results that have been derived, more thouroughly, in the *Multigrid Calculus* document. Let's proceed now with TripleGrid and make a fresh start.

Via the Cubic

The cubic equation for grid refinement with TripleGrid Calculus is:

$$y_{k+1}(y_{k+1}-3)^2 = y_k$$

$$y_{k+1}^3 - 6y_{k+1}^2 + 9y_{k+1} - y_k = 0$$

Which is of the form:

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

The discriminant of such a cubic equation is far less well known than the one of a quadratic equation. It may nevertheless be found on the Internet at:

`http://en.wikipedia.org/wiki/Discriminant`

Where we can copy and paste the following formula:

$$D = a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3 + 18a_0a_1a_2a_3 - 27a_0^2a_3^2$$

The general theory about the discriminant of a polynomial is quite interesting in itself. The abovementioned web page offers pointers to topics like i.e. the resultant and via the resultant link the Sylvester matrix of two polynomials. Whatever. Let's apply the general discriminant of the cubic to the specific one we have at hand with TripleGrid:

$$y_{k+1}^3 - 6y_{k+1}^2 + 9y_{k+1} - y_k = 0$$

With $[\ a_3 = 1 \quad a_2 = -6 \quad a_1 = 9 \quad a_0 = -y_k\ ]$ its discriminant D becomes:

$$D = 9^2(-6)^2 - 4(-y_k)(-6)^3 - 4(9)^3 1 + 18(-y_k)9(-6)1 - 27(-y_k)^2 1^2$$

$$= 3^6 2^2 - 3^6 2^2 - 2^5 3^3 y_k + 3^5 2^2 y_k - 3^3 y_k^2 = 2^2 3^3(-8+9)y_k - 3^3 y_k^2$$

$$\Longrightarrow \quad D = 27.y_k(4-y_k)$$

It is known that the above cubic equation has three real solutions if and only if this discriminant D is positive. Meaning that the following is a necessary and sufficient condition for it:

$$0 \le y_k \le 4$$

Thus, with TripleGrid coarsening, the *Dangerous Interval* becomes apparent almost immediately, from the sign of the discriminant alone.
Key reference for the rest of this paragraph is:

`http://www.sosmath.com/algebra/factor/fac111/fac111.html`

Which is based on "A new approach to solving the cubic" by R.W.D. Nickalls:

`http://www.m-a.org.uk/docs/library/2059.pdf`

Recall the cubic equation which is associated with triple grid refinement:

$$y_{k+1}^3 - 6y_{k+1}^2 + 9y_{k+1} - y_k = 0$$

It is of the form:

$$ax^3 + bx^2 + cx + d = 0$$

With $[\quad a = 1 \quad b = -6 \quad c = 9 \quad d = -y_k \quad]$. According to the above reference, we can define a quantity x_N by:

$$x_N = \frac{-b}{3a} = \frac{6}{3} = 2$$

This means that a substitution $z = y_{k+1} - 2$ or $y_{k+1} = z + 2$ will bring our cubic into its reduced form:

$$y_{k+1}^3 - 6y_{k+1}^2 + 9y_{k+1} - y_k = (z+2)^3 - 6(z+2)^2 + 9(z+2) - y_k =$$

$$z^3 + 6z^2 + 12z + 8 - 6z^2 - 24z - 24 + 9z + 18 - y_k = z^3 - 3z + 2 - y_k = 0$$

Furthermore, a quantity δ is defined by:

$$\delta^2 = \frac{b^2 - 3ac}{9a^2} = \frac{36 - 27}{9} = 1$$

Let's consider the case where $0 \le y_k \le 4$. Then there are three real solutions and we have the *Casus Irreducibilis* for the cubic equation at hand. According to the above reference, we must use a *trigonometric substitution* in order to be able to find any solutions:

$$z = 2\delta\cos(\theta) = 2\cos(\theta)$$

Upon substitution it reads:

$$8\cos^3(\theta) - 6\cos(\theta) + 2 - y_k = 0$$

In the paragraph *TripleGrid Product* we found the following formula:

$$2 + 2\cos(3\theta) = 8\cos^3(\theta) - 6\cos(\theta) + 2$$

Therefore we have to solve:

$$2 + 2\cos(3\theta) = y_k \quad \Longleftrightarrow \quad y_{k+1}(y_{k+1} - 3)^2 = y_k$$

Now define:

$$y_k = 2 + 2\cos(\phi_k) \quad \text{and} \quad y_{k+1} = 2 + 2\cos(\phi_{k+1})$$

Then we have that $\theta = \phi_{k+1}$ and:

$$y_{k+1}(y_{k+1} - 3)^2 = y_k \quad \Longleftrightarrow \quad 2 + 2\cos(3\phi_{k+1}) = 2 + 2\cos(\phi_k) \quad \Longleftrightarrow$$

$$\cos(\phi_k) = \cos(3\phi_{k+1})$$

Compare this with:

$$\cos(\phi_k) = \cos(2\phi_{k+1})$$

And it seems that we have landed on solid ground: the method of angles.

TripleGrid Refinement

The end result of the preceding paragraph is:

$$y_{k+1}(y_{k+1}-3)^2 = y_k \quad \Longleftrightarrow \quad \cos(3\phi_{k+1}) = \cos(\phi_k)$$

This gives us a very simple algorithm, which is entirely in terms of angles:

$$3\phi_{k+1} = \{ \ \phi_k \ , \ 2\pi - \phi_k \ , \ 2\pi + \phi_k \ \}$$

$$\phi_{k+1} = \left\{ \ \frac{\phi_k}{3} \ , \ \frac{2\pi - \phi_k}{3} \ , \ \frac{2\pi + \phi_k}{3} \ \right\}$$

Now we will start with the value that is certainly *dangerous*, namely $y_0 = 3$. Because that value will give rise to a division by zero during the Gaussian elimination process. Then we have:

$$\cos(\phi_0) = \frac{1}{2}y_0 - 1 = \frac{1}{2} \quad \Longrightarrow \quad \phi_0 = \frac{\pi}{3}$$

The next step is:

$$\phi_1 = \left\{ \ \frac{\pi}{9} \ , \ \frac{2\pi}{3} - \frac{\pi}{9} \ , \ \frac{2\pi}{3} + \frac{\pi}{9} \ \right\} = \pi \left\{ \ \frac{1}{9} \ , \ \frac{5}{9} \ , \ \frac{7}{9} \ \right\}$$

Add to this the initial value $\pi/3$, then we have in total for the dangerous values so far:

$$\phi_0 \cup \phi_1 = \pi \left\{ \ \frac{1}{9} \ , \ \frac{3}{9} \ , \ \frac{5}{9} \ , \ \frac{7}{9} \ \right\}$$

Each of the $\pi(1/9, 5/9, 7/9)$ may serve as an input for another sequence of solutions of our cubic. In order to bring order in the chaos, an article was posted in 'sci.math'. And Mike Guy offered a complete solution:

`http://groups.google.nl/group/sci.math/msg/331cbda22436f090?hl=nl&`

It's easier to see when you deal in integers x:

$$\phi_k = \frac{\pi \, x_{k+1}}{3^{k+1}} \quad \text{where:} \quad x_{k+1} = \{ \ x_k \ , \ 2.3^k - x_k \ , \ 2.3^k + x_k \ \}$$

Start with $x_1 = 1$, then we find for the first iterations (sorted computer output):

```
1
1 5 7
1 5 7 11 13 17 19 23 25
1 5 7 11 13 17 19 23 25 29 31 35 37 41 43 47 49 53 55 59 61 65 67 71 73 77 79
```

Quoted from his article. *There are 3^k elements of the array; they are all distinct, odd and non multiples of 3, and are all $< 3^{k+1}$.* But if we add the previous iterates to the set, each multiplied with a proper factor 3^k, then we have:

```
1
1 3 5 7
1 3 5 7 9 11 13 15 17 19 21 23 25
```

And their number is:

$$3^0 + 3^1 + 3^2 + ... + 3^k = \frac{3^{k+1} - 1}{3 - 1}$$

Which is precisely the number of odd integers between 0 and 3^{k+1}. Thus, indeed, the *cumulative distribution* of dangerous angles is given by:

$$\bigcup_{i=0}^{k} \phi_i = \frac{\pi\, x_{k+1}}{3^{k+1}} \quad \text{where:} \quad x_{k+1} = \left\{ \text{all odd numbers between 0 and } 3^{k+1} \right\}$$

The different iterates can still be distinguished, though: determine the maximum power k in 3^k by which an odd number is divisible. For example, in the set $\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25\}$, the subset $\{1, 5, 7, 11, 13, 17, 19, 23, 25\}$ belongs to iteration 3, the subset $\{3, 15, 21\}$ belongs to iteration 2, the subset $\{9\}$ belongs to iteration 1. In general, with the n-th iteration, numbers which are divisble by 3^{n-k} belong to the k-th iteration.
A slightly different setup is to start with the dangerous value $y_0 = 0$ and the corresponding angle:

$$\cos(\phi_0) = \frac{1}{2} y_0 - 1 = -1 \quad \Longrightarrow \quad \phi_0 = \pi$$

Herewith the first iteration becomes:

$$\phi_1 = \left\{ \frac{\pi}{3} \, , \, \frac{2\pi}{3} - \frac{\pi}{3} \, , \, \frac{2\pi}{3} + \frac{\pi}{3} \right\} = \pi \left\{ \frac{1}{3} \, , \, \frac{3}{3} \right\}$$

Instead of the zero'th. And the second iteration becomes:

$$\phi_2 = \left\{ \frac{\pi}{9} \, , \, \frac{2\pi}{3} - \frac{\pi}{9} \, , \, \frac{2\pi}{3} + \frac{\pi}{9} \, , \, \frac{\pi}{3} \, , \, \pi \right\} = \pi \left\{ \frac{1}{9} \, , \, \frac{3}{9} \, , \, \frac{5}{9} \, , \, \frac{7}{9} \, , \, \frac{9}{9} \right\}$$

Instead of the first. And so on and so forth. Herewith we find all odd fractions in one sweep, up and including a last one, which is equal to $3^n/3^n = 1$.
So far so good for the *Dangerous domain* with *TripleGrid Refinement*. Let the iterand y_k be in the *Safe domain*, meaning that $y_k > 4$. Then it is obvious that, with *TripleGrid Coarsening*, the iterand $y_{k+1} = y_k(y_k - 3)^2$ will again be greater than 4 and hence in the Safe domain. But is the reverse also true? Let $y_k > 4$ and solve for y_{k+1} in:

$$y_{k+1}(y_{k+1} - 3)^2 = y_k$$

Can we be certain then that $y_{k+1} > 4$ as well? Yes we can. The discriminant of this cubic, as we have seen already, is: $D = 27.y_k(4 - y_k)$. It is negative for $y_k > 4$, meaning that the corresponding cubic has two complex conjugate and one real solution. In our case, the real solution is the one that really counts. It is found by solving for the corresponding hyperbolic angle $3p_{k+1} = p_k$ and then find $y_{k+1} = 2 + 2\cosh(p_{k+1})$, which is definitely greater than 4.

Chebyshev Polynomials

Let's talk a minute about my article named *Cosine Expansions*, which is found at the location below:

`http://hdebruijn.soo.dto.tudelft.nl/hdb_spul/cospower.pdf`

Why has nobody told us that the cosine power series in that article, as well as in the current one, are actually *Chebyshev Polynomials* of the First kind?

`http://en.wikipedia.org/wiki/Chebyshev_polynomial`

And that the definition which comes most close to the employment in these two contexts is:

$$T_n(x) = \begin{cases} \cos(n \arccos(x)) & \text{for} \quad -1 \le x \le +1 \\ \cosh(n \operatorname{arccosh}(x)) & \text{for} \quad x \ge +1 \end{cases}$$

An immediate corollary is the composite identity (or the "nesting property"):

$$T_n(T_m(x)) = T_{n.m}(x)$$

Lemma. The following holds for the trigonometric as well as for the hyperbolic cosine:

$$\begin{aligned} \cos(\alpha+\beta) + \cos(\alpha-\beta) &= 2\cos(\alpha)\cos(\beta) \\ \cosh(\alpha+\beta) + \cosh(\alpha-\beta) &= 2\cosh(\alpha)\cosh(\beta) \end{aligned}$$

Proof. Add the following equations together.
For the trigonometric case:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

For the hyperbolic case:

$$\cosh(\alpha+\beta) = \cosh(\alpha)\cosh(\beta) + \sinh(\alpha)\sinh(\beta)$$
$$\cosh(\alpha-\beta) = \cosh(\alpha)\cosh(\beta) - \sinh(\alpha)\sinh(\beta)$$

Theorem.

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$$

Proof. Employ the lemma for $\alpha = n\theta$ and $\beta = \theta$:

$$\begin{aligned} \cos((n+1)\theta) + \cos((n-1)\theta) &= 2\cos(n\theta)\cos(\theta) \\ \cosh((n+1)\theta) + \cosh((n-1)\theta) &= 2\cosh(n\theta)\cosh(\theta) \end{aligned}$$

And substitute $\theta = \arccos(x))$ or $\theta = \operatorname{arccosh}(x))$ respectively. This gives:

$$\begin{aligned} \cos((n+1)\arccos(x)) + \cos((n-1)\arccos(x)) &= 2x\cos(n\arccos(x)) \\ \cosh((n+1)\operatorname{arccosh}(x)) + \cosh((n-1)\operatorname{arccosh}(x)) &= 2x\cosh(n\operatorname{arccosh}(x)) \end{aligned}$$

In short:

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x) \quad \Longleftrightarrow \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

The latter is exactly the standard recursion formula for Chebyshev Polynmials. Its initial values are:

$$\begin{aligned} T_0(x) &= \cos[h](0 \operatorname{arccos}[h](x)) = 1 \\ T_1(x) &= \cos[h](1 \operatorname{arccos}[h](x)) = x \end{aligned}$$

Herewith we find, for the Chebyshev Polynomials up to order 5:

$$\begin{aligned} T_2(x) &= 2xx - 1 = 2x^2 - 1 \\ T_3(x) &= 2x(2x^2 - 1) - x = 4x^3 - 3x \\ T_4(x) &= 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 8x^2 + 1 \\ T_5(x) &= 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x) = 16x^5 - 20x^3 + 5x \end{aligned}$$

Now we could define Han de Bruijn's Polynomials $B_n(y)$ by:

$$B_n(y) = 2T_n(y/2 - 1) + 2$$

Herewith we find:

$$\begin{aligned} B_0(y) &= 2 + 2 = 4 \\ B_1(y) &= 2(y/2 - 1) + 2 = y \\ B_2(y) &= 4(y/2 - 1)^2 - 2 + 2 = (y - 2)^2 \\ B_3(y) &= 8(y/2 - 1)^3 - 6(y/2 - 1) + 2 = (y - 2)^3 - 3(y - 2) + 2 = y(y - 3)^2 \end{aligned}$$

Thus the DoubleGrid and TripleGrid Coursening iterands are described by the polynomials $B_2(y)$ and $B_3(y)$ respectively.

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch.