DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN The Netherlands

UNDERSTANDING AND USING BROUWER'S CONTINUITY RINCI LE

Wim Veldman

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN
Toernooiveld
6525 ED Nijmegen
The Netherlands

Understanding and using Brouwer's Continuity Principle

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1 Brouwer's ontinuity Principle

$\mathbb{N}$ $, \ , \ , \ldots$

$\mathbb{N}$

$\mathcal{N}$

$\mathcal{N}$

α $\mathcal{N}$ $\mathbb{N}$ $\mathbb{N}$ α α $, \alpha$ $, \alpha$ $, \ldots$

$\mathbb{N}$

$\mathbf{0}$

$\mathbf{0}$ $\mathbf{0}$ $\mathbf{0}$ $, \ldots$

d π
d d d $, \dots$

example
n **example** n $i < n$ $j <$
d i j $j <$ d n j **example** n

example
n **example** n j
example j **example** n

example
$, \; , \; , \dots$

0

$R \subseteq \mathcal{N} \times \mathbb{N}$
$\langle \alpha, m \rangle$ R $\alpha R m$ m
α

α $\mathcal{N}$ m $\mathbb{N}$ $\alpha R m$

α $\mathcal{N}$

m $\mathbb{N}$
α m
α

α
m α m
α m
α β
α
$m, n, \dots$ $\mathbb{N}$

1.1

$R \subseteq \mathcal{N} \times \mathbb{N}$

$\forall \alpha \in \mathcal{N}\ \exists m\ \ \alpha R m$

$\forall \alpha \in \mathcal{N}\ \exists n\ \exists m\ \forall \beta \in \mathcal{N}$ $i < n$ $\alpha\ i$ $\beta\ i$ $\beta R m$

$WC - \mathbb{N}$

2 The continuity of real functions

2.1 X $\mathcal{N}$ X

s $\langle s$ $, \dots , s\ n -$ $\rangle$

α X $i < n$ $\alpha\ i$ $s\ i$

α $\mathcal{N}$ n $\mathbb{N}$ β X $i < n$

$\alpha\ i$ $\beta\ i$ α X

$\mathcal{N}$

2.2

$X \subseteq \mathcal{N}$ $R \subseteq X \times \mathbb{N}$

$\forall \alpha \in X\ \exists m\ \ \alpha R m$

$\forall \alpha \in X\ \exists n\ \exists m\ \forall \beta \in X$ $i < n$ $\alpha\ i$ $\beta\ i$ $\beta R m$

$\mathcal{N}$ X

r $\mathcal{N}$ X α X $r\ \alpha$ α

2.3 $q\ ,q\ ,\ldots$ $\mathbb{Q}$

α $\mathcal{N}$

$n\ \ |q\ \ -q\ \ |<$ —— $\mathbb{R}$

$\mathbb{R}$

2.4 $\mathbb{R}$

α,β α

α β $n\ \ |\alpha\ n\ \ -\beta\ n\ |\ <$ ——

$\mathbb{R}$

α,β n ——

γ α

2.9 b , b , . . . $\mathbb{Q}$
α i k $b \leq \alpha$ i
$b \leq \alpha$ i k α i $\leq b$ i α i k $\leq b$ i
α α $\mathbb{R}$
γ α γ

2.10 Theorem:
$\mathbb{Q}$

$\mathbb{Q}$

3 Strong counter-examples

$P \vee \neg P$
example **0** $\neg$ **example**
0 **example** **0** $\vee \neg$ **example** **0**
example **0** $\vee \neg$ **example** **0**

P $\neg\neg$ $P \vee \neg P$

α $\mathcal{N}$ α **0** $\neg$ α **0**

3.1 Theorem:
$\neg\forall\alpha \in \mathcal{N}$ α **0** $\vee \neg$ α **0**

$\forall\alpha \in \mathcal{N}$ α **0**$\vee$ / α **0**
n
α $\mathcal{N}$ $i < n$ α i α **0**
α $\mathcal{N}$ $i < n$ α i $\neg$ α **0**

3.2 Theorem:
f n f —
f

t $\mathbb{N} \to \mathbb{N}$ n q —
β
n $i \leq n$ **example** i β n t n
$i < n$ **example** i / β n t i i i
example i /
example **0** β f β **example** / **0**
f β
f β
f β $>$ f β $<$
α $\mathcal{N}$
β α **0** f β α / **0** f β
f β $>$ f β $<$ α **0** α / **0**

3.3

, $\mathbb{R}$

4 Brouwer's first application

4.1 Theorem:
$\mathcal{N}$ $\mathbb{N}$

f $\mathcal{N}$ $\mathbb{N}$
n α $\mathcal{N}$ $i < n$ α i f α f **0**

f

4.2

f $\mathbb{N} \to \mathcal{N}$ α $\mathcal{N}$ n
f n $/$ α α n α n f n n

4.3

X, Y X Y Y X
X Y X Y
Y X
$\mathbb{N}$
$\mathcal{N}$

4.4

$\mathbb{N}$ $*$

T α $\mathcal{N}$, ,

T

$$\mathbf{0},\ \langle\ \rangle * \mathbf{0},\ \langle\ ,\ \rangle * \mathbf{0},\ \langle\ ,\ ,\ \rangle * \mathbf{0},\ \ldots$$

$\mathbb{N}$ T
example T

T
T $\mathbb{N}$
T $\mathbb{N}$ T

4.5 s $\mathbb{N}$ X $\mathcal{N}$ $s * X$ $\{s * \alpha \mid \alpha \in X\}$
X, Y $\mathcal{N}$ $X \oplus Y$ $\langle\ \rangle * X \cup \langle\ \rangle * Y$ X
$\mathcal{N}$ $\cdot X$ $\cdot X$ $\cdot X, \ldots$ $\mathcal{N}$ $\cdot X$ $\emptyset$
n n $\cdot X$ $n \cdot X$ $\oplus X$ α $\mathcal{N}$ n $\mathbb{N}$
$\overline{\alpha}$ n $\langle \alpha$ $, \ldots \alpha$ $n -$ $\rangle$ $\overline{\alpha} n$
$\overline{\alpha}$ n

4.6 Theorem:

$n \quad n \cdot T \quad n \quad \cdot T$

$T \quad T \oplus T \quad T \quad T \oplus T$

$f \quad T \oplus T \quad T \quad f\ \mathbf{0} \quad \mathbf{0}$

$m \quad f\ \mathbf{0} \quad m \quad n \quad \alpha \quad T$

$i < n \quad \alpha\ i \quad f\ \alpha \quad m \quad f\ \mathbf{0} \quad m \quad f\ \alpha \quad f\ \mathbf{0} \quad f$

$f\ \langle\ \rangle * \mathbf{0} \quad \mathbf{0} \quad f$

$n \quad n \cdot T \quad n \quad \cdot T \quad n$

$f \quad n \quad \cdot T \quad n \cdot T$

$f \quad \langle\ \rangle * \mathbf{0}, \ldots, \overline{\mathbf{0}} n * \langle\ \rangle * \mathbf{0}$

$\langle\ \rangle * \mathbf{0}, \ldots, \overline{\mathbf{0}}\ n - \quad * \langle\ \rangle * \mathbf{0}$

4.7 $X \quad \mathcal{N} \quad \overline{X} \quad \alpha \quad \mathcal{N}$

$n \quad \mathbb{N} \quad \beta \quad X \quad \overline{\beta} n \quad \overline{\alpha} n$

$\overline{X} \quad X$

$\overline{X} \quad s \quad \mathbb{N} \quad \alpha$

$X \quad n \quad \overline{\alpha} n \quad s$

$T\ , T\ , T\ \underline{, \ldots} \quad \mathcal{N}$

$T \quad \{\mathbf{0}\} \quad m \quad T \quad \overline{\mathbf{0}} n * \langle\ \rangle * T$

$T \quad T \quad T$

$m \quad T \quad \alpha \quad \mathcal{N}$

$, \quad m \quad n, m \quad n \cdot T$

T

4.8 Theorem:

$n, m \quad n \cdot T \quad n \quad \cdot T \quad T \quad .$

$T \quad T \ \oplus T$

$T \quad T \ \oplus T \quad n \quad n \cdot T \quad T$

$n \cdot T \quad T$

$f \quad T \ \oplus T \quad T \quad f\ \mathbf{0} \quad \mathbf{0}$

$m \quad \overline{f\ \mathbf{0} \quad m} \quad n \quad \alpha \quad T$

$i < n \quad \alpha\ i \quad f\ \alpha \quad m \quad \overline{f\ \mathbf{0}} \quad m$

$g \quad T \quad \overline{\mathcal{N}} \quad \beta \quad T \quad f\ \beta \quad \gamma$

$f\ \overline{\mathbf{0}} n * \beta \quad \overline{f\ \mathbf{0}} \quad m \quad * \gamma \quad g \quad T \quad T$

$f\ \langle\ \rangle * \mathbf{0} \quad \mathbf{0} \quad f$

4.9

$\omega \quad \omega \quad \cdot p \quad \omega \quad \cdot$

$p \quad \cdots \quad \omega \quad \cdot p \quad \langle n\ , n\ , \ldots , n \quad \rangle$

$\langle p\ , p\ , \ldots , p \quad \rangle$

k

$\mathcal{N}$ $\mathbb{T}$ ω $\cdot p$ ω $\cdot p$ $\cdots$

ω $\cdot p$ p $\cdot T$ $\oplus p$ $\cdot T$ $\oplus \ldots \oplus p$ $\cdot T$ $<$

ω

4.10 Theorem:

α, β ω $\alpha < \beta$ $\mathbb{T}$ α $\mathbb{T}$ β

4.11

U $, U$ $, \ldots$ n U U

V $\overline{\mathbf{0}}$ n $* \langle\ \rangle * U$ n U V $n \cdot V$ n $\cdot V$

5 A model-theoretic observation

5.1

X, Y X Y

φ $\langle X \rangle \mid$ φ $\langle Y \rangle \mid$ $\neg\varphi$

$\mathbb{N}$ $\mathcal{N}$ $\forall x \forall y$ x $y \vee \neg$ x

y

T $\mathcal{N}$

α T α $\langle\ \rangle * \mathbf{0}$ α $\langle\ \rangle * \mathbf{0}$

β $\mathcal{N}$ α $\mathcal{N}$ α β α β $\exists x$ $\forall y$

x $y \vee \neg$ x y $\langle T \rangle$ $\langle \mathcal{N} \rangle$

5.2 Theorem:

α, β ω $\alpha < \beta$ $\mathbb{T}$ α $\mathbb{T}$ β

$,$ $, \ldots$

x $\neg\forall x$ x x $\vee \neg$ x x n

x x $\wedge \neg\forall x$ x $\rightarrow$ x x $\vee \neg$ x x

x, x $, x$ $, \ldots$

φ φ x $\exists$ x φ x

$\exists x$ φ x $\wedge \forall y$ φ y $\rightarrow y$ x

$\exists$ x x $\langle T$ $\rangle$ $\langle$ $\cdot T$ $\rangle$

$\exists x$ $\exists y$ x $/$ $y \wedge$ x $\wedge$ y $\wedge \forall z$ z $\rightarrow$ z $x \vee z$ y

$\langle$ $\cdot T$ $\rangle$ $\langle$ $\cdot T$ $\rangle$ n α T

T n

$\exists x$ x T T

6 Beginning the Borel hierarchy

6.1 X $\mathbb{R}$ X
q $\mathbb{Q}$ n $\mathbb{N}$ X $\{\alpha|\alpha \in \mathbb{R}|\ |\alpha - q| < -\}$
X $\mathbf{\Sigma}$ $X\ , X\ , \ldots$
X X
X $\mathbf{\Pi}$ Y X $\mathbb{R}\backslash Y$
X X

X $\mathbf{\Sigma}$ $X\ , X\ , \ldots$
X X

X $\mathbf{\Pi}$ $X\ , X\ , \ldots$
X X

$\mathbf{\Pi}$ $\mathbf{\Sigma}$

6.2 **Rat** α
q n $|q \quad - q| < -$
PosIrr α n q $<$
q $< q$ $< q$ $q \ < q$ q $< q$
α **PosIrr** q $\mathbb{Q}$ n $|\alpha - q| > -$
α

PosIrr
PosIrr
Rat $\mathbf{\Sigma}$
$\mathbf{\Pi}$ $\mathbb{R}$

6.3 Theorem:
$G\ , G\ , \ldots$
Rat $\subseteq$ G α **PosIrr** $\alpha \in$ G

$G\ , G\ , \ldots$ **Rat** $\subseteq$ G

α n
q $< q$ $< q$ $< q$ n

$\alpha\ , \alpha\ , \ldots , \alpha\ \ n -\ , \alpha\ \ n -$ q $< q$
q q
q $< q$ $< q$ $< q$ $q \ < q$ q $< q$

q α $, q$ G q $- q$ G $<$ —

6.4 Theorem:

F $, F$ $, \ldots$
PosIrr $\subseteq$ F α **Rat** $\alpha \in$ F

F $, F$ $, \ldots$ **PosIrr** $\subseteq$ F
α **PosIrr** **PosIrr**
m, n α **PosIrr** $\overline{\alpha}$ m
$\overline{\alpha}$ m α F
q $, q$ F
q $, q$ F F

6.5

$\boldsymbol{\Sigma}$ $\boldsymbol{\Pi}$

6.6 Theorem:
$,$ $\cup$ $,$ $\boldsymbol{\Sigma}$ $\boldsymbol{\Pi}$

$,$ $\cup$ $,$ $\boldsymbol{\Sigma}$ $\boldsymbol{\Pi}$
G $, G$ $, \ldots$ $,$ $\cup$ $,$ G

n $,$ $\subseteq G$ $,$ $,$ $\cup$ $,$ α
$\mathbb{R}$ $\alpha \leq$ $\leq \alpha$
α $\mathbb{R}$ n α n $--$
n α $\mathbb{R}$ $\overline{\alpha}$ n $\overline{\alpha} n$
$\alpha \leq$ α $\mathbb{R}$ $\overline{\alpha}$ n $\overline{\alpha} n$ $\leq \alpha$

6.7

7 Borel hierarchy theorems

7.1 $\mathcal{N}$ X $\mathcal{N}$
s $\mathbb{N}$ X α $\mathcal{N}$

n $\overline{\alpha}n$ s $\mathcal{N}$
$\mathbb{R}$

7.2 X, Y $\mathcal{N}$ X Y
f $\mathcal{N}$ $\mathcal{N}$ α α X
f α Y
X Y
f $\mathcal{N}$ $\mathcal{N}$ f X Y f $\mathcal{N}\backslash X$
Y

7.3 Theorem:
P Q
Q P

7.4

P h
$\mathcal{N}$ $\mathcal{N}$ P h
P, Q h , h $\mathcal{N}$ $\mathcal{N}$
P h Q h f $\mathcal{N}$
$\mathcal{N}$ P Q

$$\begin{array}{ccc} \mathcal{N} & \cdots\rightarrow & \mathcal{N} \\ h & & h \\ \mathcal{N} & \longrightarrow & \mathcal{N} \end{array}$$

α β f h α h β
g $\mathcal{N}$ $\mathcal{N}$
h $\circ g$ $f \circ h$

f P Q

7.5 $\mathcal{C}$ α $\mathcal{N}$,
Finite **Almostfinite**
Finite α $\mathcal{C}$ $\exists n\, \forall m > n$ α m
Almostfinite α $\mathcal{C}$
γ n α γ n α
Finite α $\mathbb{N}$

7.6 Theorem:

P Q
Q P $\mathbf{Finite} \subseteq Q \subseteq \mathbf{Almostfinite}$

Finite **Almostfinite**
Almostfinite **Notnotfinite** α $\mathcal{C}$
$\neg\neg\ \alpha \in$ **Finite**

Almostfinite

7.7 α, β $\mathcal{C}$ α β
n α $\overline{\beta}n$

Share T α $\mathcal{C}$
T T $\overline{\mathbf{Share}\ T}$
α $\mathcal{C}$ α $\overline{\mathbf{Share}\ T}$ n γ T
α $\overline{\gamma}n$ α $\overline{\mathbf{Share}\ T}$ $\exists n$ α $\overline{\mathbf{0}}n$ $/$
$\neg\exists n$ α $\overline{\mathbf{0}}n$ $/$ α **Share** T $\neg\neg\ \alpha \in \mathbf{Share}\ T$
$\overline{\mathbf{Share}\ T}$ **Share** T **Share** T
C

7.8 Theorem:
P Q
Q P $\mathbf{Share}\ T \subseteq Q \subseteq \overline{\mathbf{Share}\ T}$

Q P **Share** T
Almostfinite $\mathbf{\Sigma}$ **Share** T
$\mathbf{\Pi}$

References

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