Steiner Ellipses and Variances

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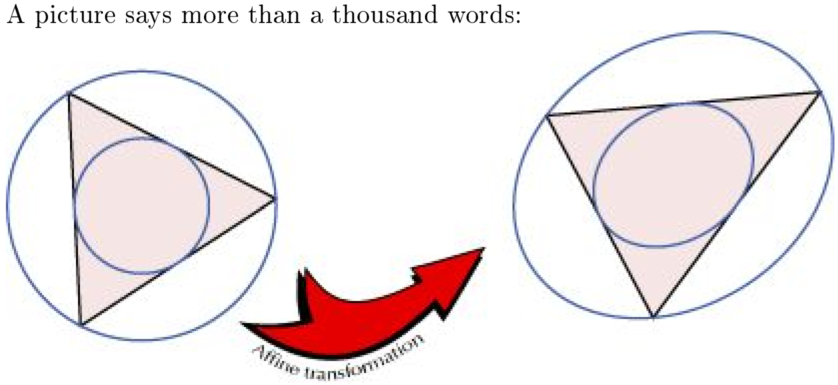
According to standard mathematics, *two* so-called Steiner ellipses are associated with an arbitrary triangle: the Circumellipse and the Inellipse.

http://mathworld.wolfram.com/SteinerEllipse.html

A picture says more than a thousand words:



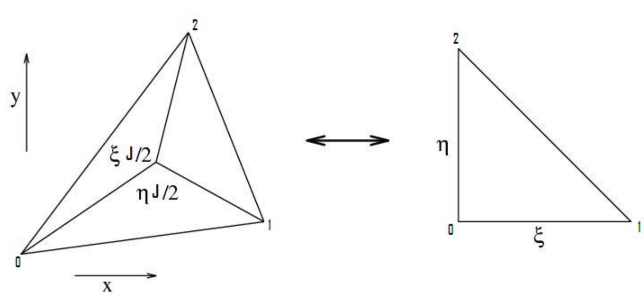
It is argued in this article that, instead of just two, a whole *family* of Steiner ellipses can be associated with a triangle. All of these are a multiple of a single ellipse, which is the so-called Ellipse of Inertia / Ellipse of Variances.

Triangle Algebra

Let's consider the simplest non-trivial finite element shape in two dimensions: the linear triangle. Function behaviour is approximated inside such a triangle by a *linear* interpolation between the function values at the vertices, also called: nodal points. Let T be such a function, and x, y coordinates, then:

$$T = A.x + \quad .y + C$$

Where the constants A, B, C are yet to be determined.



Substitute $x = x_k$ and $y = y_k$ with $k = 0, 1, 2$:

$$\begin{bmatrix} T_0 \\ T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix} \begin{bmatrix} C \\ A \\ \; \end{bmatrix}$$

The first of these equations can already be used to eliminate the constant C, once and forever:

$$T_0 = A.x_0 + \quad .y_0 + C$$

Resulting in:

$$T - T_0 = A.(x - x_0) + \quad .(y - y_0)$$

Hence the constants A and   are determined by:

$$\begin{aligned} T_1 - T_0 &= A.(x_1 - x_0) + \quad .(y_1 - y_0) \\ T_2 - T_0 &= A.(x_2 - x_0) + \quad .(y_2 - y_0) \end{aligned}$$

Two equations with two unknowns. The solution is found by straightforward elimination, or by applying Cramer's rule:

$$\begin{aligned} A &= [(y_2 - y_0).(T_1 - T_0) - (y_1 - y_0).(T_2 - T_0)]/J \\ &= [(x_1 - x_0).(T_2 - T_0) - (x_2 - x_0).(T_1 - T_0)]/J \end{aligned}$$

There are several forms of the determinant J, which should be memorized when it is appropriate:

$$\begin{aligned} J &= (x_1 - x_0).(y_2 - y_0) - (x_2 - x_0).(y_1 - y_0) \\ J &= 2 \times \text{ area of triangle} \\ J &= x_0.y_1 + x_1.y_2 + x_2.y_0 - y_0.x_1 - y_1.x_2 - y_2.x_0 \\ J &= x_0.(y_1 - y_2) + x_1.(y_2 - y_0) + x_2.(y_0 - y_1) \\ J &= y_0.(x_2 - x_1) + y_1.(x_0 - x_2) + y_2.(x_1 - x_0) \\ J &= \left\| \begin{matrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{matrix} \right\| \end{aligned}$$

Anyway, it is concluded that:

$$T - T_0 = \xi.(T_1 - T_0) + \eta.(T_2 - T_0)$$

Where:

$$\begin{aligned} \xi &= [(y_2 - y_0).(x - x_0) - (x_2 - x_0).(y - y_0)]/J \\ \eta &= [(x_1 - x_0).(y - y_0) - (y_1 - y_0).(x - x_0)]/J \end{aligned}$$

Or:

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} +(y_2 - y_0) & -(x_2 - x_0) \\ -(y_1 - y_0) & +(x_1 - x_0) \end{bmatrix} /J \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

The inverse of the following problem is recognized herein:

$$\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = \begin{bmatrix} (x_1 - x_0) & (x_2 - x_0) \\ (y_1 - y_0) & (y_2 - y_0) \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

Or:

$$\begin{aligned} x - x_0 &= \xi.(x_1 - x_0) + \eta.(x_2 - x_0) \\ y - y_0 &= \xi.(y_1 - y_0) + \eta.(y_2 - y_0) \end{aligned}$$

But also:

$$T - T_0 = \xi.(T_1 - T_0) + \eta.(T_2 - T_0)$$

Therefore the *same* expression holds for the function T as well as for the coordinates x and y . This is precisely what people mean by an *isoparametric* ("same parameters") transformation, a terminology which is quite common in Finite Element contexts. Now recall the formulas which express ξ and η into x and y :

$$\begin{aligned} \xi &= [(y_2 - y_0).(x - x_0) - (x_2 - x_0).(y - y_0)]/J \\ \eta &= [(x_1 - x_0).(y - y_0) - (y_1 - y_0).(x - x_0)]/J \end{aligned}$$

Thus ξ can be interpreted as: area of the sub-triangle spanned by the vectors $(x-x_0, y-y_0)$ and (x_2-x_0, y_2-y_0) divided by the whole triangle area. And η can be interpreted as: area of the sub-triangle spanned by the vectors $(x-x_0, y-y_0)$ and $(x_1 - x_0, y_1 - y_0)$ divided by the whole triangle area. This is the reason why ξ and η are sometimes called *area-coordinates*; see the above figure, where (two times) the area of the triangle as a whole is denoted as J. There exist even *three* of these coordinates in literature. But the third area-coordinate is, of course, dependent on the other two, being equal to $(1 - \xi - \eta)$. Instead of area-coordinates, we prefer to talk about *local coordinates* ξ and η of an element, in contrast to the *global coordinates* x and y. It is possible that local coordinates coincide with the global coordinates. A triangle for which such is the case is called a *parent element*. The portrait of the parent triangle is also depicted in the above figure: it is rectangular, and two sides of it are equal.
In the next subsection, midpoints of triangles play an important role. So let's investigate what the isoparametric transformation is for the center of gravity of a triangle. The latter is defined as:

$$\begin{cases} \overline{x} = (x_0 + x_1 + x_2)/3 \\ \overline{y} = (y_0 + y_1 + y_2)/3 \end{cases}$$

It follows that:

$$\begin{bmatrix} \overline{x} - x_0 \\ \overline{y} - y_0 \end{bmatrix} = \begin{bmatrix} (x_0 + x_1 + x_2)/3 - x_0 \\ (y_0 + y_1 + y_2)/3 - y_0 \end{bmatrix} = \begin{bmatrix} (x_1 - x_0)/3 + (x_2 - x_0)/3 \\ (y_1 - y_0)/3 + (y_1 - y_0)/3 \end{bmatrix} =$$

$$\begin{bmatrix} (x_1 - x_0) & (x_2 - x_0) \\ (y_1 - y_0) & (y_2 - y_0) \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} \quad \Longleftrightarrow$$

$$\begin{bmatrix} x - \overline{x} \\ y - \overline{y} \end{bmatrix} = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} - \begin{bmatrix} \overline{x} - x_0 \\ \overline{y} - y_0 \end{bmatrix} =$$

$$\begin{bmatrix} (x_1-x_0) & (x_2-x_0) \\ (y_1-y_0) & (y_2-y_0) \end{bmatrix} \left\{ \begin{bmatrix} \xi \\ \eta \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} \right\} \quad \Longleftrightarrow$$

$$\begin{bmatrix} x-\overline{x} \\ y-\overline{y} \end{bmatrix} = \begin{bmatrix} (x_1-x_0) & (x_2-x_0) \\ (y_1-y_0) & (y_2-y_0) \end{bmatrix} \begin{bmatrix} \xi - 1/3 \\ \eta - 1/3 \end{bmatrix}$$

The last formula must be memorized, because it will be used later on.

Triangle Integrals

In order to be able to carry out certain integrations for arbitrary triangles, it turns out that it is sufficient to calculate the following rather special integral:

$$F(m,n) = \iint \xi^m \eta^n \, d\xi d\eta$$

Herewith the integration is carried out for a parent triangle only, with local coordinates ξ and η, where $0 \le \xi \le 1$ and $0 \le \eta \le (1-\xi)$. Working out a few steps:

$$F(m,n) = \iint \xi^m \eta^n \, d\xi d\eta = \int_0^1 \xi^m \left[\int_0^{1-\xi} \eta^n d\eta \right] d\xi$$

$$= \int_0^1 \xi^m \left[\frac{(1-\xi)^{n+1}}{n+1} \right] d\xi = \int_0^1 \frac{(1-\xi)^{n+1}}{n+1} d\left(\frac{\xi^{m+1}}{m+1} \right)$$

$$= \left[\frac{(1-\xi)^{n+1}}{n+1} \frac{\xi^{m+1}}{m+1} \right]_0^1 - \int_0^1 \frac{\xi^{m+1}}{m+1} d\left(\frac{(1-\xi)^{n+1}}{n+1} \right) = 0 + \int_0^1 \frac{\xi^{m+1}}{m+1} (1-\xi)^n \, d\xi$$

$$= \frac{n}{m+1} \int_0^1 \xi^{m+1} \left[\frac{(1-\xi)^n}{n} \right] d\xi = \frac{n}{m+1} \int_0^1 \xi^{m+1} \left[\int_0^{1-\xi} \eta^{n-1} d\eta \right] d\xi$$

$$= \frac{n}{m+1} \iint \xi^{m+1} \eta^{n-1} d\xi d\eta = \frac{n}{m+1} F(m+1, n-1)$$

Now we can set up the following sequence of formulas:

$$F(m,n) = \frac{n}{m+1} F(m+1,n-1) = \frac{n}{m+1} \frac{n-1}{m+2} F(m+2,n-2) = \ldots$$

$$= \frac{n(n-1)\ldots 2.1}{(m+1)(m+2)\ldots(m+n-1)(m+n)} F(m+n,0)$$

$$= \frac{n(n-1)\ldots 2.1 \, . \, m(m-1)\,..\,2.1}{1.2.\ldots(m-1)m(m+1)\ldots(m+n)} F(m+n,0) = \frac{m!\,n!}{(m+n)!} F(m+n,0)$$

So only integrals of the form $F(m+n,0)$ are left to be calculated:

$$\iint \xi^{m+n} \, d\xi d\eta = \int_0^1 \xi^{m+n} \left[\int_0^{1-\xi} d\eta \right] d\xi = \int_0^1 \xi^{m+n} (1-\xi) d\xi$$

$$= \int_0^1 \xi^{m+n} d\xi - \int_0^1 \xi^{m+n+1} d\xi = \left[\frac{\xi^{m+n+1}}{m+n+1}\right]_0^1 - \left[\frac{\xi^{m+n+2}}{m+n+2}\right]_0^1$$
$$= \frac{1}{m+n+1} - \frac{1}{m+n+2} = \frac{(m+n+2)-(m+n+1)}{(m+n+1)(m+n+2)}$$
$$= \frac{1}{(m+n+1)(m+n+2)} = F(m+n,0)$$

Hence:

$$F(m,n) = \frac{m!\,n!}{(m+n)!}\frac{1}{(m+n+1)(m+n+2)} = \frac{m!\,n!}{(m+n+2)!}$$

This is the final result:

$$\iint \xi^m \eta^n \, d\xi d\eta = \frac{m!\,n!}{(m+n+2)!}$$

Triangle Moments

Our goal is to calculate some (zero, first and second order) moments of an arbitrary triangle. To be mathematically precise:

$$\mu_x = \frac{\iint x\,dx\,dy}{\iint dx\,dy} \quad \text{and} \quad \mu \ = \frac{\iint y\,dx\,dy}{\iint dx\,dy}$$

$$\sigma_{xx} = \frac{\iint x^2\,dx\,dy}{\iint dx\,dy} \quad \text{and} \quad \sigma \ = \frac{\iint y^2\,dx\,dy}{\iint dx\,dy} \quad \text{and} \quad \sigma_x \ = \frac{\iint xy\,dx\,dy}{\iint dx\,dy}$$

Following the theory in the previous paragraphs, global coordinates (x,y) can be expressed into their local counterparts (ξ,η) :

$$x - x_0 = \xi(x_1 - x_0) + \eta(x_2 - x_0)$$
$$y - y_0 = \xi(y_1 - y_0) + \eta(y_2 - y_0)$$

It makes no difference for the outcome of the integrals if a more handsome choice for the coordinate system is to be preferred. Therefore, let one of the vertices of the triangle, say (x_0,y_0), be temporarily selected as the origin of our global coordinate system: $(x_0,y_0) = (0,0)$. Then:

$$x = \xi x_1 + \eta x_2 \quad \text{and} \quad y = \xi y_1 + \eta y_2$$

The Jacobian J of the transformation is involved as:

$$dx\,dy = (x_1 y_2 - x_2 y_1)\,d\xi\,d\eta = J\,d\xi\,d\eta$$

This is the final result of the preceding subsection:

$$\iint \xi^m \eta^n \, d\xi\,d\eta = \frac{m!\,n!}{(m+n+2)!}$$

So let's calculate a few of these triangle moments.

$$\text{Area} = \iint dx\,dy = \iint d\xi\,d\eta\,J = \frac{0!\,0!}{(0+0+2)!}\,J = J/2 = \frac{1}{2}(x_1 y_2 - x_2 y_1)$$

Since all (other) moments have to be divided by this area, the outcome of their integrals have to be multiplied with a factor $2/J$. A first order moment is:

$$\frac{\iint x\,dx\,dy}{\text{Area}} = 2/J \iint (x_1\xi + x_2\eta)\,d\xi\,d\eta\,J = 2x_1 \iint \xi\,d\xi\,d\eta + 2x_2 \iint \eta\,d\xi\,d\eta$$

$$= 2x_1 \frac{1!\,0!}{(1+0+2)!} + 2x_2 \frac{0!\,1!}{(0+1+2)!} = x_1\,2/6 + x_2\,2/6 = (x_1 + x_2)/3$$

In very much the same way (replace x by y) we can prove that:

$$\frac{\iint y\,dx\,dy}{\text{Area}} = (y_1 + y_2)/3$$

Different though it seems, this is just the familiar result that the coordinates of the midpoint of a triangle equal one-third of the coordinates of the vertices:

$$\begin{aligned}\overline{x} &= x_0 + 1/3\,[(x_1 - x_0) + (x_2 - x_0)] = (x_0 + x_1 + x_2)/3 \\ \overline{y} &= y_0 + 1/3\,[(y_1 - y_0) + (y_2 - y_0)] = (y_0 + y_1 + y_2)/3\end{aligned}$$

Second order moments are:

$$2/J \iint x^2\,dx\,dy \quad \text{and} \quad 2/J \iint y^2\,dx\,dy \quad \text{and} \quad 2/J \iint xy\,dx\,dy$$

It is sufficient to calculate only the last integral. Proper substitutions in $\overline{xy}$ will take care of the other two later on.

$$2/J \iint xy\,dx\,dy = 2 \iint (x_1\xi + x_2\eta)(y_1\xi + y_2\eta)\,d\xi\,d\eta$$

$$= 2x_1y_1 \iint \xi^2\,d\xi\,d\eta + 2x_1y_2 \iint \xi\eta\,d\xi\,d\eta + 2x_2y_1 \iint \xi\eta\,d\xi\,d\eta + 2x_2y_2 \iint \eta^2\,d\xi\,d\eta$$

$$= 2x_1y_1 \frac{2!\,0!}{(2+0+2)!} + 2x_1y_2 \frac{1!\,1!}{(1+1+2)!} + 2x_2y_1 \frac{1!\,1!}{(1+1+2)!} + 2x_2y_2 \frac{0!\,2!}{(0+2+2)!}$$

$$= 2x_1y_1.1/12 \;+\; 2x_1y_2.1/24 \;+\; 2x_2y_1.1/24 \;+\; 2x_2y_2.1/12$$

The result is:

$$\overline{xy} = (2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2)/12$$

However, second order moments should be evaluated preferrably with respect to the midpoint:

$$\overline{xy} - \overline{x}\,\overline{y} = (2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2)/12 - (x_1 + x_2)/3\,(y_1 + y_2)/3$$

$$= (6x_1y_1 - 4x_1y_1 + 6x_2y_2 - 4x_2y_2 + 3x_1y_2 - 4x_1y_2 + 3x_2y_1 - 4x_2y_1)/36$$

$$= (2x_1y_1 + 2x_2y_2 - x_1y_2 - x_2y_1)/36$$
$$= [\, x_1y_1 + x_2y_2 + (x_2 - x_1)(y_2 - y_1)\,]\,/36$$

For an arbitrary origin $(x_0, y_0) \neq (0, 0)$ it reads:

$$\sigma_x \quad = [\, (x_1 - x_0)(y_1 - y_0) + (x_2 - x_0)(y_2 - y_0) + (x_2 - x_1)(y_2 - y_1)\,]\,/36$$

Substitute x instead of y herein:

$$\sigma_{xx} = \left[\, (x_1 - x_0)^2 + (x_2 - x_0)^2 + (x_2 - x_1)^2 \,\right]/36$$

Likewise for y instead of x:

$$\sigma \quad = \left[\, (y_1 - y_0)^2 + (y_2 - y_0)^2 + (y_2 - y_1)^2 \,\right]/36$$

Thus the variances σ_{xx} and σ are indeed both positive, as they should be. The following abbreviations are introduced:

$$\begin{cases} s_{xx} = (x_1 - x_0)^2 + (x_2 - x_0)^2 + (x_2 - x_1)^2 \\ s_x \;\; = (x_1 - x_0)(y_1 - y_0) + (x_2 - x_0)(y_2 - y_0) + (x_2 - x_1)(y_2 - y_1) \\ s \quad\, = (y_1 - y_0)^2 + (y_2 - y_0)^2 + (y_2 - y_1)^2 \end{cases}$$

It is thus seen that the moments of inertia of a triangle are:

$$\begin{bmatrix} \sigma_{xx} & \sigma_x \\ \sigma_x & \sigma \end{bmatrix} = \begin{bmatrix} s_{xx} & s_x \\ s_x & s \end{bmatrix} /36$$

Well known *invariants* of such a tensor (matrix) are the *determinant* and the *trace*. We compute both for the tensor on the right hand side, apart from the constant 36. As far as the trace is concerned, it is trivial that it is indeeed invariant for (orthogonal coordinate) transformations:

$$s_{xx} + s \quad = \text{ sum of the squares of the edges of the triangle}$$

The deteriminant $(s_{xx}s \quad - s_x^2\ )$ is somewhat more difficult to elaborate. We shall employ MAPLE for this purpose.

```
> x10 := x1-x0; x20 := x2-x0; x12 := x1-x2;
> y10 := y1-y0; y20 := y2-y0; y12 := y1-y2;
> sxy := x10*y10 + x20*y20 + x12*y12;
> sxx := x10*x10 + x20*x20 + x12*x12;
> syy := y10*y10 + y20*y20 + y12*y12;
> vgl := (y10*x20 - x10*y20)^2
       + (y10*x12 - x10*y12)^2
       + (y20*x12 - x20*y12)^2;
> evalb(simplify(syy*sxx-sxy^2)=simplify(vgl));
```

true

The *vgl* variable in MAPLE has not been chosen by coincidence, of course. All the terms in that variable are the square of a determinant. And all of the determinants are the areas of paralellograms. But each paralellogram area J is twice the area of the triangle. Thus all of the terms must be equal. Hence:

$$s_{xx}s \quad - s_x^2 \;\; = 3\left[\,(y_1 - y_0)(x_2 - x_0) - (x_1 - x_0)(y_2 - y_0)\,\right]^2 = 3\,J^2$$

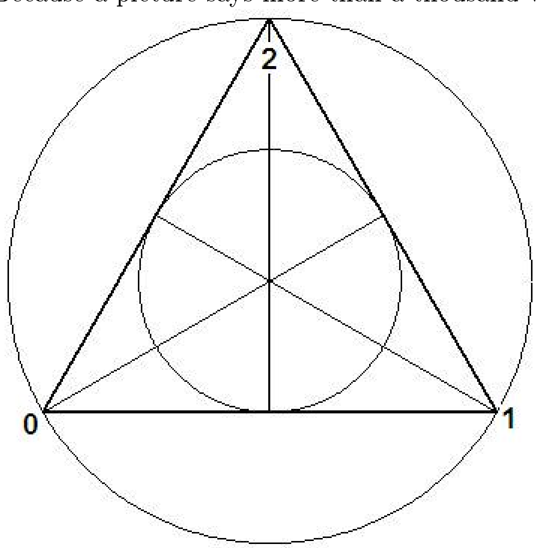
Exercise. With some knowledge of linear algebra, it must be possible to arrive at a simpler derivation.

Special Steiner Ellipses

Let us start with a equilateral triangle, which is transformed into an arbitrary triangle. This is done in two steps. The first step is to transform the equilateral triangle into the parent rectangular triangle. We suppose that both the inner and outer circles have the origin $(0,0)$ as the midpoint and the edges of the equilaterial triangle have length $= 1$. The coordinates of the equilateral triangle are given by:

$$\begin{cases} (x_0, y_0) = (-\frac{1}{2}, -\frac{1}{2}\sqrt{3}/3) \\ (x_1, y_1) = (+\frac{1}{2}, -\frac{1}{2}\sqrt{3}/3) \\ (x_2, y_2) = (0, \sqrt{3}/3) \end{cases}$$

Because a picture says more than a thousand words:



The radius of the inner circle is thus $\sqrt{3}/6$ and the radius of the outer circle is $\sqrt{3}/3$. So the equations of the inner and outer circle are, respectively:

$$x^2 + y^2 = \frac{1}{12} \qquad ; \qquad x^2 + y^2 = \frac{1}{3}$$

The coordinate transformation is:

$$\begin{bmatrix} x+1/2 \\ y+\sqrt{3}/6 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \quad \Longleftrightarrow$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} - \begin{bmatrix} 1/2 \\ \sqrt{3}/6 \end{bmatrix} \quad \Longleftrightarrow$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \xi + \eta/2 - 1/2 \\ \eta\sqrt{3}/2 - \sqrt{3}/6 \end{bmatrix}$$

And the circles thus become ellipses. It is anticipated that the midpoint of these ellipses is the midpoint of the parent triangle again, which in the (ξ,η) coordinate system is $(1/3,1/3)$.

$$x^2+y^2 = (\xi+\eta/2-1/2)^2 + (\eta\sqrt{3}/2-\sqrt{3}/6)^2 =$$

$$(\xi^2+\eta^2/4+\xi\eta+1/4-\xi-\eta/2) + (3\eta^2/4-\eta/2+1/12) =$$

$$\xi^2+\eta^2+\xi\eta-\xi-\eta+1/3 = (\xi-1/3)^2+(\xi-1/3)(\eta-1/3)+(\eta-1/3)^2$$

So the equations of the inner and outer ellipses are:

$$\left(\xi-\frac{1}{3}\right)^2 + \left(\xi-\frac{1}{3}\right)\left(\eta-\frac{1}{3}\right) + \left(\eta-\frac{1}{3}\right)^2 = R^2$$

Where:

$$R = \sqrt{\left(\frac{1}{12} \quad \text{or} \quad \frac{1}{3}\right)}$$

A parameter representation of these ellipses is found from the fact that

$$\left\{ \begin{array}{l} x = R\cos(t) \\ y = R\sin(t) \end{array} \right. \quad \Longrightarrow \quad x^2+y^2=R^2$$

Together with the inverse of the above transformation:

$$\begin{bmatrix} x+1/2 \\ y+\sqrt{3}/6 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \quad \Longrightarrow$$

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{1}{\sqrt{3}/2} \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x+1/2 \\ y+\sqrt{3}/6 \end{bmatrix} = \begin{bmatrix} 1 & -1/\sqrt{3} \\ 0 & 2/\sqrt{3} \end{bmatrix} \begin{bmatrix} x+1/2 \\ y+\sqrt{3}/6 \end{bmatrix}$$

$$= \begin{bmatrix} (x+1/2)-1/\sqrt{3}(y+\sqrt{3}/6) \\ 2/\sqrt{3}(y+\sqrt{3}/6) \end{bmatrix} = \begin{bmatrix} x-y/\sqrt{3}+1/3 \\ 2y/\sqrt{3}+1/3 \end{bmatrix} \quad \Longrightarrow$$

$$\begin{bmatrix} \xi-1/3 \\ \eta-1/3 \end{bmatrix} = \begin{bmatrix} x-y/\sqrt{3} \\ 2y/\sqrt{3} \end{bmatrix} = R \begin{bmatrix} \cos(t)-\sin(t)/\sqrt{3} \\ 2\sin(t)/\sqrt{3} \end{bmatrix}$$

General Steiner Ellipses

The time has come for the second step: transforming the parent triangle into an arbitrary triangle. The center of gravity formula found is to be employed:

$$\begin{bmatrix} x - \overline{x} \\ y - \overline{y} \end{bmatrix} = \begin{bmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{bmatrix} \begin{bmatrix} \xi - 1/3 \\ \eta - 1/3 \end{bmatrix} \quad \Longrightarrow$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} + \begin{bmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{bmatrix} R \begin{bmatrix} \cos(t) - \sin(t)/\sqrt{3} \\ 2\sin(t)/\sqrt{3} \end{bmatrix}$$

The parameter representation of the ellipses, needed for *drawing* them, is thus found immediately. The inverse of this transformation is needed too:

$$\begin{bmatrix} \xi - 1/3 \\ \eta - 1/3 \end{bmatrix} = \frac{1}{J} \begin{bmatrix} y_2 - y_0 & -(x_2 - x_0) \\ -(y_1 - y_0) & x_1 - x_0 \end{bmatrix} \begin{bmatrix} x - \overline{x} \\ y - \overline{y} \end{bmatrix}$$

Where the Jacobian $J = (x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)$ is twice the area of the triangle. Let's substitute this into the equations of the Steiner ellipses:

$$\left(\xi - \frac{1}{3}\right)^2 + \left(\xi - \frac{1}{3}\right)\left(\eta - \frac{1}{3}\right) + \left(\eta - \frac{1}{3}\right)^2 = R^2 \quad \Longrightarrow$$

$$[\ (y_2 - y_0)(x - \overline{x}) - (x_2 - x_0)(y - \overline{y})\ ]^2 +$$
$$[\ (y_2 - y_0)(x - \overline{x}) - (x_2 - x_0)(y - \overline{y})\ ][\ (x_1 - x_0)(y - \overline{y}) - (y_1 - y_0)(x - \overline{x})\ ]$$
$$+ [\ (x_1 - x_0)(y - \overline{y}) - (y_1 - y_0)(x - \overline{x})\ ]^2 = R^2 J^2$$

For the sake of simplicity, the following substitutions are adopted:

$$\begin{cases} x_{10} = x_1 - x_0 \quad ; \quad y_{10} = y_1 - y_0 \\ x_{20} = x_2 - x_0 \quad ; \quad y_{20} = y_2 - y_0 \end{cases}$$

Giving an abbreviation of the above and nothing else:

$$(y_{20}x' - x_{20}y')^2 + (y_{20}x' - x_{20}y')(x_{10}y' - y_{10}x') + (x_{10}y' - y_{10}x') = R^2 J^2$$

Where $x' = x - \overline{x}$ and $y' = y - \overline{y}$. The primes $'$ will readily be dropped for the sake of simplicity. Anyway, collecting terms with (x^2, xy, y^2) is easier now:

$$(y_{20}^2 - y_{20}y_{10} + y_{10}^2)\, x^2 +$$
$$(-2y_{20}x_{20} + y_{20}x_{10} + x_{20}y_{10} - 2x_{10}y_{10})\, xy +$$
$$(x_{20}^2 - x_{20}x_{10} + x_{10}^2)\, y^2 = R^2 J^2$$

Or:

$$\frac{1}{2}\left[\, y_{20}^2 + y_{10}^2 + (y_{10} - y_{20})^2 \,\right] x^2$$

$$-2\,\frac{1}{2}\left[\, y_{20}x_{20} + y_{10}x_{10} + (y_{10} - y_{20})(x_{10} - x_{20}) \,\right] x\, y$$

$$+\frac{1}{2}\left[\, x_{20}^2 + x_{10}^2 + (x_{10} - x_{20})^2 \,\right] y^2 = R^2 J^2$$

Substitute back the original variables $x \to x - \overline{x}$ and $y \to y - \overline{y}$. This results in:

$$\left[\, (y_2 - y_0)^2 + (y_1 - y_0)^2 + (y_1 - y_2)^2 \,\right] (x - \overline{x})^2$$
$$-2\,[\, (y_2 - y_0)(x_2 - x_0) + (y_1 - y_0)(x_1 - x_0) + (y_1 - y_2)(x_1 - x_2) \,]\, (x - \overline{x})(y - \overline{y})$$
$$+ \left[\, (x_2 - x_0)^2 + (x_1 - x_0)^2 + (x_1 - x_2)^2 \,\right] (y - \overline{y})^2 = 2\, R^2 J^2$$

The moments of inertia from *Triangle Moments* are recognized herein:

$$(x_1 - x_0)(y_1 - y_0) + (x_2 - x_0)(y_2 - y_0) + (x_2 - x_1)(y_2 - y_1) = s_x$$
$$(x_1 - x_0)^2 + (x_2 - x_0)^2 + (x_2 - x_1)^2 = s_{xx}$$
$$(y_1 - y_0)^2 + (y_2 - y_0)^2 + (y_2 - y_1)^2 = s$$
$$s_{xx} s \quad - s_x^2 \;\; = 3\, J^2$$

Hence the Steiner ellipses can be presented in the following form.

$$s \quad (x - \overline{x})^2 - 2 s_x \;\, (x - \overline{x})(y - \overline{y}) + s_{xx}(y - \overline{y})^2 = 2\, R^2 J^2$$

Where:

$$J^2 = (s_{xx} s \quad - s_x^2 \;\,)/3$$

Repeated from *Triangle Moments*:

$$36\, \sigma_x \;\; = s_x \quad \text{and} \quad 36\, \sigma_{xx} = s_{xx} \quad \text{and} \quad 36\, \sigma \quad = s$$

Consequently:

$$\frac{\sigma \quad (x - \overline{x})^2 - 2\sigma_x \;\, (x - \overline{x})(y - \overline{y}) + \sigma_{xx}(y - \overline{y})^2}{\sigma_{xx}\sigma \quad - \sigma_x^2} = \frac{2 \times 36}{3}\, R^2 = 24\, R^2$$

Where:

$$R = \sqrt{\left(\frac{1}{12} \quad \text{or} \quad \frac{1}{3}\right)}$$

Now we have all the ingredients to jump to conclusions.

Family of Steiner Ellipses

The kernel $E(x, y)$ of the Ellipse of Inertia (of a triangle) is defined by:

$$E(x, y) = \frac{\sigma \quad (x - \mu_x)^2 - 2\sigma_x \;\, (x - \mu_x)(y - \mu \;\;) + \sigma_{xx}(y - \mu \;\;)^2}{\sigma_{xx}\sigma \quad - \sigma_x^2}$$

Where $\mu_x = \overline{x}$ and $\mu \;\; = \overline{y}$. The ellipse is thus based upon the *inverse* tensor of inertia / matrix of variances, hence its name.

$$E(x, y) = \left[\; (x - \mu_x) \quad (y - \mu \;\;) \;\right] \left[\begin{array}{cc} \sigma_{xx} & \sigma_x \\ \sigma_x & \sigma \end{array}\right]^{-1} \left[\begin{array}{c} (x - \mu_x) \\ (y - \mu \;\;) \end{array}\right]$$

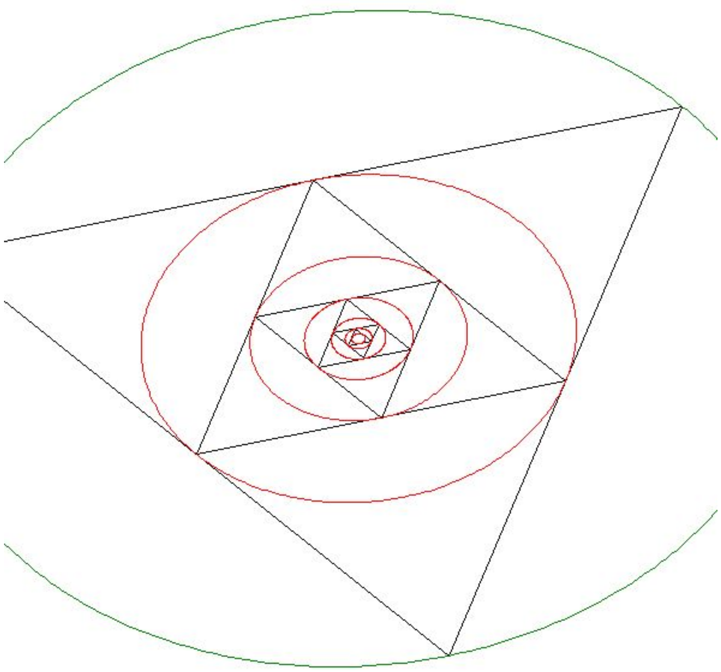
From the previous subsections, we conclude that the the Steiner Inellipse and Circumellipse are respectively given by:

$$E(x,y) = 2 \quad \text{or} \quad E(x,y) = 8$$

We might actually have a whole *family* of Steiner ellipses, defined as:

$$E(x,y) = F \qquad \text{where} \quad F > 0$$

Special cases are $F = 1$ for the Ellipse of Inertia / Variances, $F = 2$ for the Inellipse and $F = 8$ for the Circumellipse.



Another problem is whether and when an ellipse from this family could be considered as a Best Fit Ellipse. The latter shall be defined with help of a Least Squares minimization principle, as follows:

$$\sum_k w_k \left[\frac{\sigma \quad (x_k - \mu_x)^2 - 2\sigma_x \;\; (x_k - \mu_x)(y_k - \mu \;\;) + \sigma_{xx}(y_k - \mu \;\;)^2}{\sigma_{xx}\sigma \quad - \sigma_x^2} - F \right]^2$$

$= minimum(F)$. In the continuous case, such as with a triangle, summations must be replaced by integrals. Differentiating to F simply gives:

$$\sum_k w_k \left[\frac{\sigma \quad (x_k - \mu_x)^2 - 2\sigma_x \;\; (x_k - \mu_x)(y_k - \mu \;\;) + \sigma_{xx}(y_k - \mu \;\;)^2}{\sigma_{xx}\sigma \quad - \sigma_x^2} - F \right] = 0$$

If and only if:

$$\sigma \quad \sum_k w_k(x_k - \mu_x)^2 - 2\sigma_x \ \sum_k w_k(x_k - \mu_x)(y_k - \mu \ ) + \sigma_{xx} \sum_k w_k(y_k - \mu \ )^2$$

$$= F(\sigma_{xx}\sigma \quad - \sigma_x^2 \ )$$

If and only if:

$$\sigma \quad \sigma_{xx} - 2\sigma_x \ \sigma_x \ + \sigma_{xx}\sigma \quad = 2\,(\sigma_{xx}\sigma \quad - \sigma_x^2 \ ) = F(\sigma_{xx}\sigma \quad - \sigma_x^2 \ )$$

If and only if $F = 2$. Thus the equation of the Best Fit Ellipse is:

$$E(x, y) = 2$$

For a triangle, this is precisely the Steiner Inellipse. We conclude that the Inellipse is, at the same time, a Best Fit Ellipse.

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch :-)