Steiner Ellipsoid for Hexahedron

Author: Han de Bruijn
Dated: 2011 March

It is argued in this article that a Steiner Ellipsoid can be associated with a Hexahedron. To be precise: it is the CircumEllipsoid of the dual polyhedron of that hexahedron (which happens to be an Octahedron). Gauss Continuization can thus be defined in three dimensions as well. Before you start complaining about unknown terminology, here is some prerequisite reading:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2011/steiners.pdf`
`http://hdebruijn.soo.dto.tudelft.nl/jaar2011/gauss_2d.pdf`
`http://hdebruijn.soo.dto.tudelft.nl/jaar2011/steiner2.pdf`

And last but not least, my article on *Numerical Method for 3D Ideal Flow*:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2010/octaeder.pdf`

Theory

The 3-D result which is of immediate use for us is found on page 10 of *Numerical Method for 3D Ideal Flow*. The key formula is:

$$f_o(\xi,\eta,\zeta) = f_{o0} + (f_{o2} - f_{o0})\,\xi + (f_{o4} - f_{o0})\,\eta + (f_{o6} - f_{o0})\,\zeta$$

Specialized for the space vector $\vec{r}_o$ with components (x_o, y_o, z_o):

$$\begin{bmatrix} x_o(\xi,\eta,\zeta) - x_{o0} \\ y_o(\xi,\eta,\zeta) - y_{o0} \\ z_o(\xi,\eta,\zeta) - z_{o0} \end{bmatrix} = \begin{bmatrix} x_{o2} - x_{o0} \\ y_{o2} - y_{o0} \\ z_{o2} - z_{o0} \end{bmatrix} \xi + \begin{bmatrix} x_{o4} - x_{o0} \\ y_{o4} - y_{o0} \\ z_{o4} - z_{o0} \end{bmatrix} \eta + \begin{bmatrix} x_{o6} - x_{o0} \\ y_{o6} - y_{o0} \\ z_{o6} - z_{o0} \end{bmatrix} \zeta$$

It means that a vector $\vec{r}_o$ in the *dual octahedron* of an arbitrary hexahedron, with (x_{o0}, x_{o0}, x_{o0}) as the origin, is a linear combination of certain *base vectors*, defined as follows.

$$\vec{a} = \begin{bmatrix} x_{o2} - x_{o0} \\ y_{o2} - y_{o0} \\ z_{o2} - z_{o0} \end{bmatrix} \quad ; \quad \vec{b} = \begin{bmatrix} x_{o4} - x_{o0} \\ y_{o4} - y_{o0} \\ z_{o4} - z_{o0} \end{bmatrix} \quad ; \quad \vec{c} = \begin{bmatrix} x_{o6} - x_{o0} \\ y_{o6} - y_{o0} \\ z_{o6} - z_{o0} \end{bmatrix}$$

Assume that $(x_{o0}, x_{o0}, x_{o0}) = (0,0,0) = \vec{0}$. Then:

$$\vec{r}_o = \xi\,\vec{a} + \eta\,\vec{b} + \zeta\,\vec{c}$$

Additional facts:

$$\begin{bmatrix} x_{o1} - x_{o0} \\ y_{o1} - y_{o0} \\ z_{o1} - z_{o0} \end{bmatrix} = -\vec{a} \quad ; \quad \begin{bmatrix} x_{o3} - x_{o0} \\ y_{o3} - y_{o0} \\ z_{o3} - z_{o0} \end{bmatrix} = -\vec{b} \quad ; \quad \begin{bmatrix} x_{o5} - x_{o0} \\ y_{o5} - y_{o0} \\ z_{o5} - z_{o0} \end{bmatrix} = -\vec{c}$$

The base vectors are collected in a matrix :

$$= \begin{bmatrix} a_x & b_x & c_x \\ a & b & c \\ a_z & b_z & c_z \end{bmatrix}$$

Quite analogous to the theory which was developed for the two dimensional case in *auss-Steiner for Quadrilaterals*, we have the definition of first and second order moments (of iniertia) or variances. The first order moments lead to the already defined common midpoint of the hexahedron and octahedron, which has been chosen as our local origin. The second order moments of the octahedron - conceived namely as an obvious generalization of the 2-D case - are defined as follows.

$$\begin{cases} \sigma_{xx} = a_x^2 + b_x^2 + c_x^2 \\ \sigma_x \;\; = a_x a \;\, + b_x b \;\, + c_x c \\ \sigma_{xz} = a_x a_z + b_x b_z + c_x c_z \\ \sigma \;\;\;\, = a^2 + b^2 + c^2 \\ \sigma_{\;z} = a \;\, a_z + b \;\, b_z + c \;\, c_z \\ \sigma_{zz} = a_z^2 + b_z^2 + c_z^2 \end{cases}$$

And collected in a matrix Σ:

$$\Sigma = \begin{bmatrix} \sigma_{xx} & \sigma_x & \sigma_{xz} \\ \sigma_x & \sigma & \sigma_{\;z} \\ \sigma_{xz} & \sigma_{\;z} & \sigma_{zz} \end{bmatrix}$$

Theorem. The determinant of the matrix of variances is the square of the determinant of the matrix of base vectors.

$$\begin{vmatrix} \sigma_{xx} & \sigma_x & \sigma_{xz} \\ \sigma_x & \sigma & \sigma_{\;z} \\ \sigma_{xz} & \sigma_{\;z} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} a_x & b_x & c_x \\ a & b & c \\ a_z & b_z & c_z \end{vmatrix}^2 \quad \Longleftrightarrow \quad \det(\Sigma) = \det(\quad)^2$$

Theorem. A Steiner Ellipsoid can be defined for an arbitrary hexahedron as the ellipsoid that goes through the vertices of the dual polyhedron of the hexahedron (the latter is an octahedron). Define the function $E(x, y, z) =$

$$\begin{bmatrix} (x - \mu_x) & (y - \mu \;) & (z - \mu_z) \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_x & \sigma_{xz} \\ \sigma_x & \sigma & \sigma_{\;z} \\ \sigma_{xz} & \sigma_{\;z} & \sigma_{zz} \end{bmatrix}^{-1} \begin{bmatrix} x - \mu_x \\ y - \mu \\ z - \mu_z \end{bmatrix}$$

Then the Steiner ellipsoid of the hexa/octahedron has the form $E(x, y, z) = 1$. It may come as a surprise that there is no restriction whatsoever on the shape of the hexahedron. It may be even completely scrambled! That doesn't matter, because the *dual* polyhedron is very *neat*, when compared with the hexahedral finite element it is derived from.
At last, we have Gauss Continuization with the Steiner ellipsoid functions $E_h(x, y, z)$ of hexahedrons h. This should be a piece of cake now:

$$\overline{f}(x, y, z) = \sum_h \frac{e^{-E_h(x,\;\,,z)}}{\pi^{3/2}\sqrt{\det(\Sigma_h)}} \overline{f}_h J_h$$

Here $\overline{f}_h$ are mean values of a function f, discret(iz)e(d) at the vertices of any hexahedron/octahedron and J_h is the volume of the hehahedron h. A proper estimate for the expansion factor α might be derived from:

$$\sigma_{xx}^2 + \sigma^2 \quad + \sigma_{zz}^2 =$$

$$a_x^2 + b_x^2 + c_x^2 + a^2 + b^2 + c^2 + a_z^2 + b_z^2 + c_z^2 =$$

$$|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$$

Where the base vectors each are elongated with a factor α, defined as:

$$\alpha = \frac{\sqrt{2\ln(2/\epsilon)}}{\pi}$$

An admissible error ϵ is given. It follows that $\sqrt{\det(\Sigma)}$, which is equal to $|\det(\quad)|$, the absolute value of the volume spanned by the base vectors, must be multiplied by α^3. A factor α^2 must be augmented with the denominator in the Gaussian exponent - the latter can easily be inferred from the fact that the ellipsoid in standard (eigenvalues) form is $x^2/\sigma_{xx} + y^2/\sigma \quad + z^2/\sigma_{zz} = 1$. So this is the end-result:

$$\overline{f}(x,y,z) = \sum_h \frac{e^{-E_h(x,\ ,z)/\alpha^2}}{\alpha^3 \pi^{3/2} \sqrt{\det(\Sigma_h)}} \overline{f}_h J_h$$

Numerical confirmation and source of all software is supplied with the article:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2011/steiner3.zip`

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch :-)