Gauss-Steiner for Quadrilaterals

Author: Han de Bruijn
Dated: 2011 March

According to standard mathematics, *two* so-called Steiner ellipses are associated with an arbitrary triangle: the Circumellipse and the Inellipse.

`http://mathworld.wolfram.com/SteinerEllipse.html`

It is argued in this article that a Steiner ellipse can also be associated with a Quadrilateral. To be precise: it is the CircumEllipse of the dual polygon of that quadrilateral (which happens to be a paralellogram). As with the triangle, the Steiner ellipse is always a multiple of the Ellipse of Inertia / Ellipse of Variances. Gauss Continuization is much simpler than with triangles. Before you start complaining about unknown terminology, here is some prerequisite reading:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2011/steiners.pdf`
`http://hdebruijn.soo.dto.tudelft.nl/jaar2011/gauss_2d.pdf`

Drawing an Ellipse

An ellipse can be conceived as an affine transformation of a circle.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mu_x \\ \mu \end{bmatrix} + \begin{bmatrix} R \\ 0 \end{bmatrix} \cos(t) + \begin{bmatrix} 0 \\ R \end{bmatrix} \sin(t)$$

$$\begin{bmatrix} R \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} a_x \\ a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ R \end{bmatrix} \rightarrow \begin{bmatrix} b_x \\ b \end{bmatrix}$$

Indeed, when making a drawing of an ellipse, it would be handsome to have it in parametrized form. That is, we seek an equivalent like this:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mu_x \\ \mu \end{bmatrix} + \begin{bmatrix} a_x \\ a \end{bmatrix} \cos(t) + \begin{bmatrix} b_x \\ b \end{bmatrix} \sin(t)$$

$$\Longleftrightarrow \quad \begin{cases} x = \mu_x + a_x \cos(t) + b_x \sin(t) \\ y = \mu \ \ + a \ \ \cos(t) + b \ \ \sin(t) \end{cases}$$

Multiply the first equation with a , the second with a_x and substract:

$$a \ \ (x - \mu_x) - a_x(y - \mu \ \ ) = (b_x a \ \ - a_x b \ \ ) \sin(t) \quad \Longrightarrow$$

$$\sin(t) = \frac{a \ \ (x - \mu_x) - a_x(y - \mu \ \ )}{b_x a \ \ - a_x b}$$

Multiply the first equation with b , the second with b_x and substract:

$$b \ \ (x - \mu_x) - b_x(y - \mu \ \ ) = (b \ \ a_x - a \ \ b_x) \cos(t) \quad \Longrightarrow$$

$$\cos(t) = \frac{b\ \ (x - \mu_x) - b_x(y - \mu\ \ )}{b\ \ a_x - a\ \ b_x}$$

Now use the well known identity:

$$\cos^2(t) + \sin^2(t) = 1$$

Giving:

$$\left(\frac{a\ \ x' - a_x y'}{b\ \ a_x - a\ \ b_x}\right)^2 + \left(\frac{b\ \ x' - b_x y'}{b_x a\ \ - a_x b\ \ }\right)^2 = 1$$

Where $x' = x - \mu_x$ and $y' = y - \mu\ \ $. Drop the primes $'$ for the sake of simplicity. To put it otherwise, choose the midpoint of the ellipse as the origin of the coordinate system: $(\mu_x, \mu\ \ ) = (0, 0)$.

$$\left(\frac{a\ \ x - a_x y}{b\ \ a_x - a\ \ b_x}\right)^2 + \left(\frac{b\ \ x - b_x y}{b_x a\ \ - a_x b\ \ }\right)^2 = 1$$

And work out:

$$\frac{(a^2 + b^2)\ x^2 - 2(a_x a\ \ + b_x b\ \ )\ xy + (a_x^2 + b_x^2)\ y^2}{(b\ \ a_x - a\ \ b_x)^2} = 1$$

Or:

$$(a^2 + b^2)\ x^2 - 2(a_x a\ \ + b_x b\ \ )\ xy + (a_x^2 + b_x^2)\ y^2 = (b\ \ a_x - a\ \ b_x)^2$$

Check this as well:

$$(a_x^2 + b_x^2)(a^2 + b^2) - (a_x a\ \ + b_x b\ \ )^2 =$$

$$a_x^2 a^2 + a_x^2 b^2 + b_x^2 a^2 + b_x^2 b^2 - (a_x^2 a^2 + b_x^2 b^2 + 2 a_x a\ \ b_x b\ \ ) =$$

$$(a_x b\ \ )^2 + (a\ \ b_x)^2 - 2(a_x b\ \ )(a\ \ b_x) = (a_x b\ \ - a\ \ b_x)^2$$

Define the quantities $(A, \quad, C)$ as follows

$$\begin{cases} A = (a^2 + b^2) \\ \quad = (a_x a\ \ + b_x b\ \ ) \\ C = (a_x^2 + b_x^2) \end{cases}$$

Then it follows that:

$$(AC - \frac{1}{4}\quad ^2) = (a^2 + b^2)(a_x^2 + b_x^2) - (a_x a\ \ + b_x b\ \ )^2 = (a_x b\ \ - a\ \ b_x)^2$$

The following *standard form* of the ellipse is herewith suggested:

$$\frac{Ax^2 - 2\quad xy + Cy^2}{AC - \frac{1}{4}\quad ^2} = 1$$

Indeed. Suppose that we have another member of the same family of ellipses:

$$\frac{Ax^2 - 2\ \ xy + Cy^2}{AC - \frac{1}{4}\ \ ^2} = F$$

With $F > 0$ an arbitrary positive constant. Then:

$$\frac{F.Ax^2 - 2F.\ \ xy + F.Cy^2}{(AC - \frac{1}{4}\ \ ^2).F^2} = 1$$

Simply re-define the quantities $(A, \ \ , C)$ as follows and we're done.

$$\begin{cases} A = F.(a^2 + b^2) \\ \quad = F.(a_x a \ \ + b_x b \ \ ) \\ C = F.(a_x^2 + b_x^2) \end{cases}$$

With $(\mu_x, \mu \ \ ) = (0, 0)$ as the origin, the parameter representation still is:

$$\begin{cases} x = a_x \cos(t) + b_x \sin(t) \\ y = a \ \ \cos(t) + b \ \ \sin(t) \end{cases}$$

And the derivatives are:

$$\begin{cases} dx/dt = -a_x \sin(t) + b_x \cos(t) \\ dy/dt = -a \ \ \sin(t) + b \ \ \cos(t) \end{cases}$$

The area of the ellipse is then expressed by:

$$\oint \frac{1}{2}(x\,dy - y\,dx) =$$

$$\frac{1}{2} \oint \{[a_x \cos(t) + b_x \sin(t)]\,[-a \ \ \sin(t) + b \ \ \cos(t)]\ dt$$

$$- [a \ \ \cos(t) + b \ \ \sin(t)]\,[-a_x \sin(t) + b_x \cos(t)]\ dt\} =$$

$$\frac{1}{2} \oint \{[-a_x a \ \ + b_x b \ \ + a_x a \ \ - b_x b \ \ ] \cos(t) \sin(t) +$$

$$[a_x b \ \ - b_x a \ \ ] \cos^2(t) + [a_x b \ \ - b_x a \ \ ] \sin^2(t)\}\ dt = \frac{1}{2} \int_0^{2\pi} [a_x b \ \ - b_x a \ \ ]\ 1\ dt$$

$$\Longrightarrow \quad \oint \frac{1}{2}(x\,dy - y\,dx) = \pi\,|a_x b \ \ - b_x a \ \ | = \pi \sqrt{AC - \frac{1}{4}\ \ ^2}$$

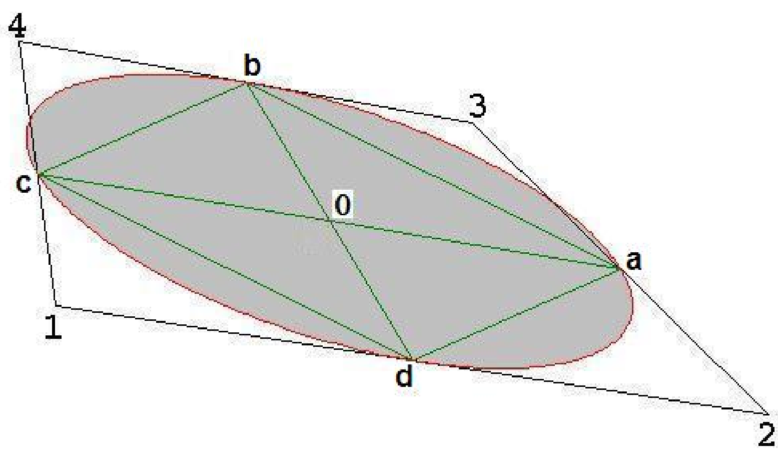
So the area of the ellipse is π times the paralellogram area spanned by the vectors $\vec{a}$ and $\vec{b}$ in its parameter representation; absolute value if a positive outcome is to be preferred. It is also equal to π times half the root of minus the (negative) discriminant of the conic section.

Steiner InEllipse

Consider an arbitrary quadrilateral, with vertex coordinates as follows.

$$\vec{1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad ; \quad \vec{2} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \quad ; \quad \vec{3} = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \quad ; \quad \vec{4} = \begin{bmatrix} x_4 \\ y_4 \end{bmatrix}$$

A picture says more than a thousand words:



Additional definitions:

$$\vec{0} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} (x_1+x_2+x_3+x_4)/4 \\ (y_1+y_2+y_3+y_4)/4 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} a_x \\ a \end{bmatrix} = \begin{bmatrix} (x_2+x_3)/2 - x_0 \\ (y_2+y_3)/2 - y_0 \end{bmatrix} = \begin{bmatrix} x_a \\ y_a \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} b_x \\ b \end{bmatrix} = \begin{bmatrix} (x_3+x_4)/2 - x_0 \\ (y_3+y_4)/2 - y_0 \end{bmatrix} = \begin{bmatrix} x_b \\ y_b \end{bmatrix}$$

Easy to prove consequences:

$$\begin{bmatrix} (x_1+x_4)/2 - x_0 \\ (y_1+y_4)/2 - y_0 \end{bmatrix} = \begin{bmatrix} -a_x \\ -a \end{bmatrix} = \begin{bmatrix} x_c \\ y_c \end{bmatrix}$$

$$\begin{bmatrix} (x_1+x_2)/2 - x_0 \\ (y_1+y_2)/2 - y_0 \end{bmatrix} = \begin{bmatrix} -b_x \\ -b \end{bmatrix} = \begin{bmatrix} x_d \\ y_d \end{bmatrix}$$

In the language of geometry: (a,b,c,d) is a *parallellogram*. The diagonals of this paralellogram are intersecting each other in equal pieces at $\vec{0}$. It is thus obvious that the following ellipse, from the previous subsection, goes through the four vertices of this paralellogram.

$$\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix} = \vec{0} + \vec{a}\cos(t) + \vec{b}\sin(t) \quad \Longleftrightarrow \quad \begin{cases} x = x_0 + a_x \cos(t) + b_x \sin(t) \\ y = y_0 + a \ \ \cos(t) + b \ \ \sin(t) \end{cases}$$

The paralellogram is the *dual polygon* of our quadrilateral. We conclude that there is no Steiner ellipse for an arbitrary quadrilateral, but there *is* always a Steiner (Circum)Ellipse for the dual polygon of a quadrilateral. The latter ellipse can be regarded as a Steiner InEllipse of the quadrilateral. (With somewhat less nice properties, perhaps, when compared with the Steiner ellipses for triangles)

Ellipse of Inertia

Let's investigate now if there is a relationship between the Steiner ellipse of the dual polygon and its *Ellipse of Inertia*. To that end, we must be able to calculate the moments of inertia / variances of the dual polygon. Conceiving it as a Quadrilateral gives one possible interpolation. Conceiving it as a Five Point Star gives another possible interpolation. We need an interpolation anyway for carrying out integrations associated with the moments. Prerequisite reading for the quadrilateral is availaible in PDF format:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2004/vierhoek.pdf`

Prerequisite reading for the Five Point Star is available only as plain TEXT:

`http://hdebruijn.soo.dto.tudelft.nl/www/article/SUNA04.NET`

But there is a shortcut via another project, available in PDF format as well:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2010/octaeder.pdf`

It's easier to work top down from this three dimensional result than to build a new theory for two dimensions, from first principles bottom up. (The latter may be a useful exercise for some. Not that it shall lead to a different insight, though.) The 3-D result which is of immediate use for us is found on page 10 of *Numerical Method for 3D Ideal Flow*:

$$f_o(\xi,\eta,\zeta) = f_{o0} + (f_{o2} - f_{o0})\,\xi + (f_{o4} - f_{o0})\,\eta + (f_{o6} - f_{o0})\,\zeta$$

Removing 3-D redundancy means to get rid of the local coordinate ζ:

$$f_o(\xi,\eta) = f_{o0} + (f_{o2} - f_{o0})\,\xi + (f_{o4} - f_{o0})\,\eta$$

Transformation for the x and y coordinates is isoparametric:

$$\begin{cases} x_o(\xi,\eta) - x_{o0} = (x_{o2} - x_{o0})\,\xi + (x_{o4} - x_{o0})\,\eta \\ y_o(\xi,\eta) - y_{o0} = (y_{o2} - y_{o0})\,\xi + (y_{o4} - y_{o0})\,\eta \end{cases}$$

The midpoint $\vec{0}$ is perferrably - and finally - adopted as the origin $(x_0, y_0) = (x_{o0}, y_{o0}) = (0,0)$ of our (x,y) coordinate system. Implementing this:

$$\begin{bmatrix} x_o(\xi,\eta) \\ y_o(\xi,\eta) \end{bmatrix} = \begin{bmatrix} x_{o2} \\ y_{o2} \end{bmatrix} \xi + \begin{bmatrix} x_{o4} \\ y_{o4} \end{bmatrix} \eta$$

At last, the following objects can safely be identified:

$$\begin{bmatrix} x_{o2} \\ y_{o2} \end{bmatrix} = \vec{a} \quad ; \quad \begin{bmatrix} x_{o4} \\ y_{o4} \end{bmatrix} = \vec{b}$$

So this is the end result, for our two dimensional case:

$$\vec{r}(\xi,\eta) = \vec{a}\,\xi + \vec{b}\,\eta \quad \Longleftrightarrow \quad \begin{cases} x(\xi,\eta) = a_x\,\xi + b_x\,\eta \\ y(\xi,\eta) = a\ \ \xi + b\ \ \eta \end{cases}$$

Another prerequisite reading is passing by:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2011/steiners.pdf`

This is the final result from "Triangle Integrals" in the above publication:

$$\iint \xi^m \eta^n \, d\xi d\eta = \frac{m!\,n!}{(m+n+2)!}$$

What we observe is that the dual polygon has a linear interpolation, which is extended over four triangles, namely

$$\left(\vec{a},\vec{b},\vec{0}\right) \quad ; \quad \left(\vec{b},\vec{c},\vec{0}\right) \quad ; \quad \left(\vec{c},\vec{d},\vec{0}\right) \quad ; \quad \left(\vec{d},\vec{a},\vec{0}\right)$$

It must be confirmed that the origin $\vec{0}$ has indeed first order moments as its components. For the first triangle:

$$\iint \left(\vec{a}\,\xi + \vec{b}\,\eta\right) dx\,dy / \iint dx\,dy = 2\vec{a}\iint \xi\,d\xi\,d\eta + 2\vec{b}\iint \eta\,d\xi\,d\eta = \frac{1}{3}\vec{a} + \frac{1}{3}\vec{b}$$

Likewise for the other triangles, giving:

$$\frac{1}{3}\left[\left(\vec{a}+\vec{b}\right) + \left(\vec{b}+\vec{c}\right) + \left(\vec{c}+\vec{d}\right) + \left(\vec{d}+\vec{a}\right)\right] =$$

$$\frac{1}{3}\left[\left(\vec{a}+\vec{b}\right) + \left(\vec{b}-\vec{a}\right) + \left(-\vec{a}-\vec{b}\right) + \left(-\vec{b}+\vec{a}\right)\right] = \vec{0}$$

Now go for the second order moments. Start with:

$$\begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix} = \begin{bmatrix} (a_x\xi + b_x\eta)^2 & (a_x\xi + b_x\eta)(a\ \ \xi + b\ \ \eta) \\ (a_x\xi + b_x\eta)(a\ \ \xi + b\ \ \eta) & (a\ \ \xi + b\ \ \eta)^2 \end{bmatrix} =$$

$$\begin{bmatrix} a_x & b_x \\ a & b \end{bmatrix} \begin{bmatrix} \xi^2 & \xi\eta \\ \xi\eta & \eta^2 \end{bmatrix} \begin{bmatrix} a_x & a \\ b_x & b \end{bmatrix}$$

So the integrals to be calculated are:

$$2\iint \xi^2 d\xi\,d\eta = 2\iint \eta^2 d\xi\,d\eta = 2\frac{2.1}{2.3.4} = \frac{1}{6}$$

$$2\iint \xi\,\eta\,d\xi\,d\eta = 2\frac{1.1}{2.3.4} = \frac{1}{12}$$

For the triangles $\left(\vec{a}, \vec{b}, \vec{0}\right)$ and $\left(-\vec{a}, -\vec{b}, \vec{0}\right)$ resulting in:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{x} \\ \sigma_{x} & \sigma \end{bmatrix} =$$

$$\frac{1}{6}\begin{bmatrix} a_x^2 + a_x b_x + b_x^2 & a_x a \; + (a_x b \; + a \; b_x)/2 + b_x b \\ a_x a \; + (a_x b \; + a \; b_x)/2 + b_x b & a^2 + a \; b \; + b^2 \end{bmatrix}$$

Add this to the results for the other triangles $\left(\vec{b}, -\vec{a}, \vec{0}\right)$ and $\left(-\vec{b}, \vec{a}, \vec{0}\right)$:

$$\frac{1}{6}\begin{bmatrix} a_x^2 - a_x b_x + b_x^2 & a_x a \; - (a_x b \; + a \; b_x)/2 + b_x b \\ a_x a \; - (a_x b \; + a \; b_x)/2 + b_x b & a^2 - a \; b \; + b^2 \end{bmatrix}$$

The end result is the weighted mean of all these values. Note, however, that the areas of the triangles are all the same. Consequently, summation is easy:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{x} \\ \sigma_{x} & \sigma \end{bmatrix} = \frac{1}{6}\begin{bmatrix} a_x^2 + b_x^2 & a_x a \; + b_x b \\ a_x a \; + b_x b & a^2 + b^2 \end{bmatrix}$$

The Ellipse of Inertia - with the orgin as the midpoint - is thus:

$$\begin{bmatrix} x & y \end{bmatrix} \frac{1}{6}\begin{bmatrix} a_x^2 + b_x^2 & a_x a \; + b_x b \\ a_x a \; + b_x b & a^2 + b^2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

Apart from the factor 1/6, this is exactly the ellipse as known from previous subsections, such as *Drawing an Ellipse*. But the latter is precisely our Steiner ellipse for the (inner paralellogram of the) quadrilateral. It is thus proved that the Steiner ellipse of a quadrilateral is six times the ellipse of inertia of the same quadrilateral:

$$\begin{bmatrix} (x-\mu_x) & (y-\mu \;) \end{bmatrix} \frac{1}{\sigma_{xx}\sigma \; - \sigma_x^2} \begin{bmatrix} \sigma & -\sigma_x \\ -\sigma_x & \sigma_{xx} \end{bmatrix} \begin{bmatrix} x-\mu_x \\ y-\mu \end{bmatrix} = 6$$

Let:

$$E(x,y) = \frac{\sigma \; (x-\mu_x)^2 - 2\sigma_x \; (x-\mu_x)(y-\mu \;) + \sigma_{xx}(y-\mu \;)^2}{\sigma_{xx}\sigma \; - \sigma_x^2}$$

Then, for the Steiner ellipse of a quadrilateral:

$$E(x,y) = 6$$

Gauss Continuization

The theory of Gauss-Steiner Continuization on quadrilaterals is quite analogous to the theory of continuization for triangles, but it is significantly simpler. In two dimensions, take a non-constant discrete function f_i, defined at the vertices (x_i, y_i) of quadrilaterals in a Finite Element mesh. The midpoints of these quads (Q) are $(\overline{x}_Q, \overline{y}_Q)$ and corresponding function values $\overline{f}_Q = (f_1 + f_2 + f_3 + f_4)/4$ (with $\overline{x}_Q$ and $\overline{y}_Q$ as special cases).

$$\overline{f}(x, y) = \sum_Q G_Q(x, y)\, \overline{f}_Q\, J_Q/2$$

Here J_Q are twice the (positive) areas of the quadrilaterals (Q). Twice the area of a quadrilateral can be calculated easily with determinants:

$$J_Q = x_1 y_2 - x_1 y_4 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_4 + x_4 y_1 - x_4 y_3$$

The distributions G_Q, for the moment being, are the following:

$$G_Q(x, y) = \frac{e^{-\frac{1}{2}\left[s_{yy}(x-\mu_x)^2 - 2s_{xy}(x-\mu_x)(\quad -\mu_y) + s_{xx}(\quad -\mu_y)^2\right]/(s_{xx}s_{yy} - s_{xy}^2)}}{2\pi\sqrt{s_{xx}s \quad - s_x^2}}$$

Where:

$$6\,\alpha^2 \begin{bmatrix} \sigma_{xx} & \sigma_x \\ \sigma_x & \sigma \end{bmatrix} = \alpha^2 \begin{bmatrix} s_{xx} & s_x \\ s_x & s \end{bmatrix} = \alpha^2 \begin{bmatrix} a_x^2 + b_x^2 & a_x b_x + a\ b \\ a_x b_x + a\ b & a^2 + b^2 \end{bmatrix}$$

Here μ and σ are the first and second order moments of the quad (Q) and α is an enlargement of the Steiner ellipse, dependent on the desired accuracy. An estimate previously employed (with triangles) is:

$$\alpha = \frac{\sqrt{2\ln(2/\epsilon)}}{\pi}$$

The denominator of G_Q can be analyzed further with results from the previous subsections. It is twice the area of the α-extended Steiner ellipse:

$$2\pi\sqrt{s_{xx}s \quad - s_x^2} = \alpha^2\, 2\pi\, |a_x b \quad - a\ \ b_x| = 2 \times (\text{ellipse area})$$

Gathering everything together, here comes the final formula. It is noted that the quadrilateral areas J_Q, in general, do *not* cancel out against the areas of the (restricted) Steiner ellipses.

$$\overline{f}(x, y) = \frac{1}{2\pi\,\alpha^2} \sum_Q \frac{J_Q/2}{|a_x b \quad - a\ \ b_x|_Q} G_Q(x, y)\, \overline{f}_Q$$

And G_Q is redefined as:

$$G_Q(x, y) = e^{-\frac{1}{2}\left[s_{yy}(x-\mu_x)^2 - 2s_{xy}(x-\mu_x)(\quad -\mu_y) + s_{xx}(\quad -\mu_y)^2\right]/(s_{xx}s_{yy} - s_{xy}^2)}$$

All the other quantites have been previously defined. There are a few issues left, such as: how many quadrilaterals have to be taken into account, in order to arrive at a sensible approximation for the function f? But it is supposed that these questions can be answered in an analogous way as with triangles, an exercise that has been done before.

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch :-)