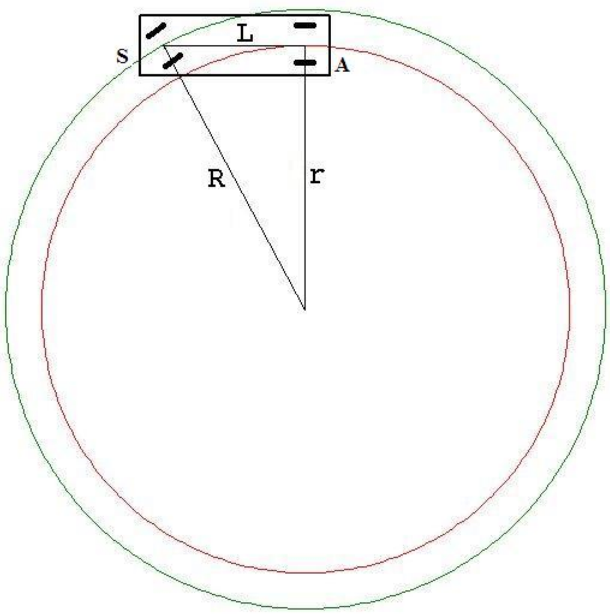
Long Vehicle Kinematics

Imagine a long vehicle with length L, riding on a traffic circle with radius R. Steering is done via the front wheels and, to be quite precise, R is the radius of the circle traversed by the midpoint of the axis joining the front wheels. Ackermann steering geometry is assumed for the front wheels:

`http://en.wikipedia.org/wiki/Ackermann_steering_geometry`

Furthermore, it is assumed that the rear wheels are fixed, though most probably provided with a differential gear. If conditions of no slip are assumed for the rear wheels, then it is obvious that the rear wheels each must be perpendicular to the radius of another circle with radius r, where r is the mean distance of the rear wheels to the midpoint of both circles.



From the above picture, it is obvious, though somewhat surprising perhaps, that the relationship between the two radii R and r and the length L of the vehicle is simply given by Pythagoras theorem, where R and L are known and r is to be calculated:

$$r^2 + L^2 = R^2 \quad \Longrightarrow \quad r = \sqrt{R^2 - L^2}$$

Differential Equations

It is questioned what the equations of motion are of the midpoint $\vec{r}_A$ between the rear wheels. There is no question about the equations of motion of the midpoint $\vec{r_S}$ between the front wheels, because it's there where the steering is. Let $\vec{L} = \vec{r_S} - \vec{r_A}$ be the vector representing the vehicle, which is the vector that joins (the middles of) the front $\vec{r_S}$ and the rear $\vec{r_A}$. Then the equations of motion are as follows, as a bit thinking shall reveal.

$$\vec{v}_A = \dot{\vec{r}}_A = \frac{(\vec{v_S} \cdot \vec{L})}{(\vec{L} \cdot \vec{L})}\vec{L} = \frac{(\vec{v_S} \cdot \vec{L})}{L^2}\vec{L}$$

In words: the velocity vector $\vec{v}_A$ of the rear wheels is the projection of the velocity vector $\vec{v}_S$ of the front wheels on the vehicle vector $\vec{L}$.
It will be demonstrated in the first place that the simple solution for a vehicle riding on a traffic circle satisfies the differential equations. Step by step and let MAPLE do the work. Abbreviations are employed for $c = \cos(\phi)$ and $s = \sin(\phi)$ where ϕ is the angle between the vector $\vec{r_S}$ to the front and the vector $\vec{r_A}$ to the rear. The solution is:

$$\left\{ \begin{array}{l} x_A(t) = r\cos(\omega t - \phi) = r\left[\cos(\omega t)\cos(\phi) + \sin(\omega t)\sin(\phi)\right] \\ y_A(t) = r\sin(\omega t - \phi) = r\left[\sin(\omega t)\cos(\phi) - \cos(\omega t)\sin(\phi)\right] \end{array} \right. \qquad \Longrightarrow$$

$$\left\{ \begin{array}{l} x_A(t) = r\left[\cos(\omega t)r/R + \sin(t)L/R\right] \\ y_A(t) = r\left[\sin(\omega t)r/R - \cos(t)L/R\right] \end{array} \right.$$

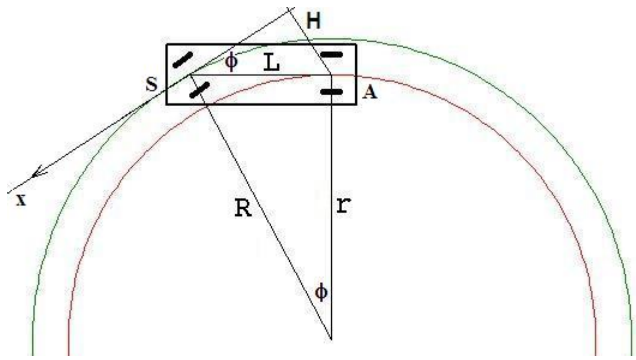
Substituting into MAPLE learns that it satisfies the differential equations:

```
> r := sqrt(R^2-L^2);
> x_S := R*cos(omega*t); y_S := R*sin(omega*t);
> c := r/R; s := L/R;
> x_A := r*(cos(omega*t)*c + sin(omega*t)*s);
> y_A := r*(sin(omega*t)*c - cos(omega*t)*s);
> L_x := x_S - x_A; L_y := y_S - y_A;
> u_S := diff(x_S,t); v_S := diff(y_S,t);
> u_A := diff(x_A,t); v_A := diff(y_A,t);
> prj := (L_x*u_S + L_y*v_S)/L^2;
> evalb(simplify(u_A = prj*L_x));

                              true

> evalb(simplify(v_A = prj*L_y));

                              true
```

Ride on Exit

Imagine the same long vehicle with length L, but with its front wheels no longer riding on that traffic circle. The differential equations are, when written in scalar form:

$$\left\{ \begin{array}{l} \dot{x}_A = \left[U.(x_S - x_A) + V.(y_S - y_A)\right]/L^2.(x_S - x_A) \\ \dot{y}_A = \left[U.(x_S - x_A) + V.(y_S - y_A)\right]/L^2.(y_S - y_A) \end{array} \right.$$



Assume that the front wheels of the vehicle are exiting the traffic circle and are now moving on a straight line. Assume the direction of the x-axis for that straight line. Then $V = 0$. Hence:

$$\dot{x}_A = U.(x_S - x_A)^2/L^2$$

The calculations are started when the front wheels are at the beginning of the straight line but the rear wheels are still riding on the traffic circle. Then, as a little geometry reveals, the latter are off the straight line at a distance $L : R = H : L \Longrightarrow H = L^2/R$. And:

$$x_S(t) = Ut + \sqrt{L^2 - H^2} \quad \Longrightarrow \quad \dot{x}_S = U \quad \Longrightarrow \quad \dot{x}_S - \dot{x}_A = U - U(x_S - x_A)^2/L^2$$

Hence the differential equation:

$$\dot{x} = U\left[1 - (x/L)^2\right] \qquad \text{where} \quad x = x_S - x_A$$

With boundary condition:

$$x_A(0) = 0 \quad \Longrightarrow \quad x(0) = \sqrt{L^2 - H^2}$$

Solve:

$$\frac{dx/L}{1 - (x/L)^2} = U/L\,dt$$

$$\frac{1}{2}\frac{d(+x/L)}{1 + x/L} - \frac{1}{2}\frac{d(-x/L)}{1 - x/L} = \frac{U}{L}dt$$

$$\frac{1}{2}\ln(1 + x/L) - \frac{1}{2}\ln(1 - x/L) = \frac{U}{L}t + C$$

$$\ln\left(\frac{1 + x/L}{1 - x/L}\right) = 2\frac{U}{L}t + C$$

$$\frac{1 - x/L}{1 + x/L} = Ce^{-2.U/L.t}$$

Lemma (without proof):

$$a = \frac{1-b}{1+b} \quad \Longleftrightarrow \quad b = \frac{1-a}{1+a}$$

Herewith:

$$\frac{x}{L} = \frac{1 - Ce^{-2.U/L.t}}{1 + Ce^{-2.U/L.t}}$$

$$x_A(t) = x_S(t) - L\,\frac{1 - Ce^{-2.U/L.t}}{1 + Ce^{-2.U/L.t}}$$

Where the constant C is yet to be determined. For $t = 0$:

$$\frac{1 - \sqrt{L^2 - H^2}/L}{1 + \sqrt{L^2 - H^2}/L} = C$$

For $t \to \infty$ (vehicle on straight road) is $x_A(t) = x_S(t) - L$.
Much more interesting, of course, is the solution for the y coordinate:

$$\dot{y}_A = \left[U.(x_S - x_A) + V.(y_S - y_A)\right]/L^2.(y_S - y_A)$$

Where $y_S(t) = 0$, $V = 0$, $x_S(t) - x_A(t) = x(t)$ and $x_A(t)$ is a known outcome.
Let $y_S(t) - y_A(t) = y(t)$. Then:

$$(y_S \dot{-} y_A)(t) = \dot{y} = -U.(x_S - x_A)/L^2.(y_S - y_A) = -U.x/L^2.y$$

$$\dot{y}/y = -U.x/L^2 \quad \Longrightarrow \quad y(t) = Ye^{-\int U.x(t)/L^2.dt}$$

With $y(0) = -H$. Here $x(t)$ is the solution found previously. I have resorted to MAPLE to do the rest of the work for me:

```
> C := (1-sqrt(L^2-H^2)/L)/(1+sqrt(L^2-H^2)/L);
> x(t) := L*(1-C*exp(-2*U/L*t))/(1+C*exp(-2*U/L*t));
> simplify(exp(-int(U*x(t)/L^2,t)));
```

The outcome is not even overly complicated:

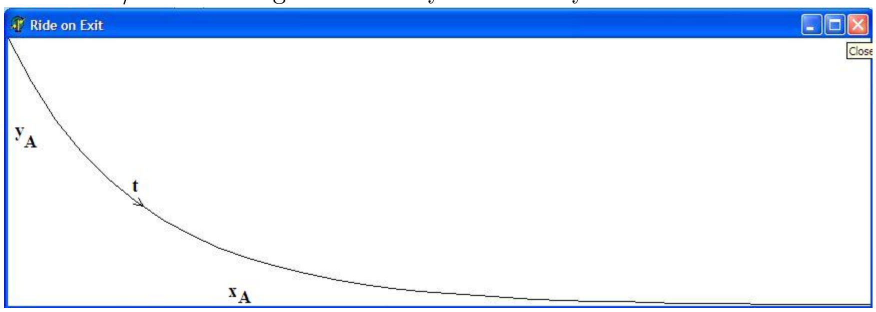
$$\frac{\sqrt{e^{\left(-\frac{2Ut}{L}\right)}}}{L + \sqrt{L^2 - H^2} + e^{\left(-\frac{2Ut}{L}\right)}L - e^{\left(-\frac{2Ut}{L}\right)}\sqrt{L^2 - H^2}}$$

For $t = 0$ we find (by hand) that $y_S(0) - y_A(0) = -H = y(0)$. So:

$$Y\,\frac{1}{L + \sqrt{L^2 - H^2} + L - \sqrt{L^2 - H^2}} = \frac{Y}{2L} = -H \quad \Longrightarrow \quad Y = -2LH = -2L^3/R$$

$$y_A(t) = \frac{2L^3/Re^{-U/Lt}}{L + \sqrt{L^2 - H^2} + e^{\left(-\frac{2Ut}{L}\right)}L - e^{\left(-\frac{2Ut}{L}\right)}\sqrt{L^2 - H^2}}$$

For $t \to \infty$ this outcome approaches zero. The *decay time* for this to happen is of order L/U = the length divided by the velocity of the vehicle. Like this:



Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch :-)