Gauss-Steiner Continuization

Author: Han de Bruijn
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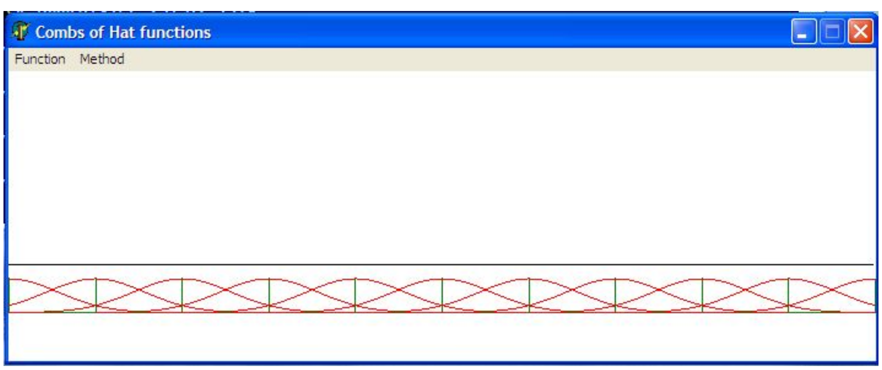
The definition of Gauss-Steiner Continuization, as employed in this article, is the following. A set of discrete real function values is the range of values to be approximated, with a function that is continuous and differentiable. This is to be accomplished with a comb of Gauss distributions. Therefore we start with a generalization of one-dimensional Uniform Combs of Gaussians, for irregular 1-D grids and non-constant functions. The two-dimensional discretization has an arbitrary Finite Element like mesh of triangles as its domain. With help of the family of Steiner ellipses, an analogue of the one-dimensional comb of Gauss distributions is constructed. The discretization at hand is made continuous and *differentiable* in this way. Prerequisite reading is the article "Steiner Ellipses and Variances" at:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2011/steiners.pdf`

Combs of Gaussians

Prerequisite reading:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2010/dikte/document.pdf`
`http://hdebruijn.soo.dto.tudelft.nl/jaar2010/kammen/document.pdf`



The Fourier series of a Uniform Comb of Gaussians $G(x)$ is given by:

$$G(x) = \sum_{L=-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-L\Delta)^2/\sigma^2} \Delta = 1 + 2 \times \sum_{k=1}^{\infty} e^{-\frac{1}{2}(k\omega\sigma)^2} cos(k\omega x)$$

Here Δ is the grid spacing, x is the one-dimensional coordinate, σ is the spread, $\omega = 2\pi/\Delta$. Suppose that the outcome of the *Poisson Summation Formula* on the right hand side is approximately the constant function $f(x) = 1$. Then it has been found that such is resulting in the following condition for the spread, if ϵ is a given tolerance:

$$|G(x) - 1| \le \epsilon \quad \Longleftrightarrow \quad \sigma \ge \frac{\Delta}{2\pi}\sqrt{2\ln(2/\epsilon)}$$

Two generalizations are possible now. The first one is a non-uniform grid. The second one is a non-constant function. We start with the first generalization. Instead of a uniform grid $L\Delta$, consider coordinates $x_0, x_1, x_2, \ldots, x_i, \ldots$, where $x_0 < x_1 < x_2 \ldots < x_i < \ldots$ Then define $\Delta_i = x_i - x_{i-1}$ for $i = 1, 2, 3, \ldots$ It is expected that the comb of Gaussians at an irregular grid will converge likewise to the constant function $f(x) = 1$:

$$\sum_i \frac{1}{\sigma_i\sqrt{2\pi}}\, e^{-\frac{1}{2}(x-x_i)^2/\sigma_i^2}\, \Delta_i \approx 1$$

Provided that irregular spreads σ_i are related to the now irregular intervals Δ_i according to:

$$\frac{\sigma_i}{\frac{1}{2}\Delta_i} \ge \alpha \qquad \text{where} \quad \alpha = \frac{\sqrt{2\ln(2/\epsilon)}}{\pi}$$

In order to avoid confusion with a quantity in the next subsection, which is the area of a triangle, we replace $\Delta_i/2$ by R_i, i.e. the *radius* of the interval. And we assume equality:

$$\sigma_i = \alpha\, R_i \qquad \text{where} \quad \alpha = \frac{\sqrt{2\ln(2/\epsilon)}}{\pi}$$

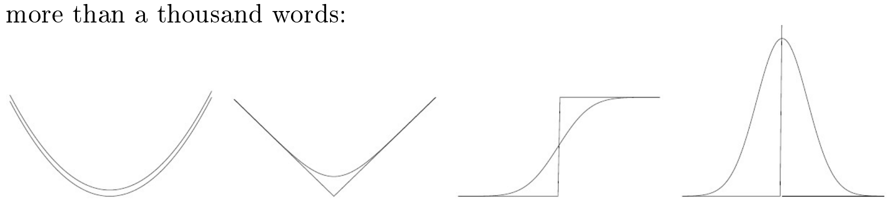
The second generalization is a non-constant discrete function, defined at the grid points $\{x_i\}$ as real values f_i. We consider the midpoints of the intervals Δ_i as $\overline{x}_i = (x_i + x_{i-1})/2$ and $\overline{f}_i = (f_i + f_{i-1})/2$ for $i > 0$. Then:

$$\sum_i \frac{1}{\sigma_i\sqrt{2\pi}}\, e^{-\frac{1}{2}(x-\overline{x}_i)^2/\sigma_i^2}\, \overline{f}_i\, \Delta_i = \overline{f}(x)$$

Where it is expected that $f(\overline{x}_i) \approx \overline{f}_i$ and therefore $\overline{f}$ can be considered as a *continuization* of the discretization f_i, ipse est an attempt to make the discrete continuous. It is noted that the above can be considered as a Riemann sum of the following integral.

$$\int_{-\infty}^{+\infty} \lim_{\sigma\to 0} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(x-\xi)^2/\sigma^2} f(\xi)\, d\xi = \int_{-\infty}^{+\infty} \delta(x-\xi) f(\xi)\, d\xi = f(x)$$

Not a quite rigorous argument for confirming what we "expect", but it will do. The grid is refined first. The spread is so much coarser that the Riemann sum is a good approximation of the integral "before" the bell shapes of the Gauss distributions approximate the delta function. Therefore it is expected that the function $\overline{f}$ is rather a *smoothing* of the "real" function f. Again, a picture says more than a thousand words:



Skewed 2-D Bell Shape

Associated with the first and second order moments in one dimension is the Gauss function, also known as the *normal distribution* in Statistics:

$$G(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} = \frac{1}{\sqrt{2\pi\sigma_{xx}}} e^{-\frac{1}{2}(x-\mu_x)^2/\sigma_{xx}}$$

The exponent (apart from the factor 1/2) could have been written as:

$$(x-\mu_x)^2/\sigma_{xx} = (x-\mu_x)\frac{1}{\sigma_{xx}}(x-\mu_x)$$

In the general two-dimensional case, σ_{xx} will be replaced by the tensor:

$$\begin{bmatrix} \sigma_{xx} & \sigma_x \\ \sigma_x & \sigma \end{bmatrix}$$

And the inverse $1/\sigma_{xx}$ by the inverse of this matrix:

$$\begin{bmatrix} \sigma_{xx} & \sigma_x \\ \sigma_x & \sigma \end{bmatrix}^{-1} = \begin{bmatrix} \sigma & -\sigma_x \\ -\sigma_x & \sigma_{xx} \end{bmatrix} / (\sigma_{xx}\sigma \quad - \sigma_x^2 \ )$$

The accompanying quadratic form is:

$$\begin{bmatrix} (x-\mu_x) & (y-\mu \ ) \end{bmatrix} \begin{bmatrix} \sigma & -\sigma_x \\ -\sigma_x & \sigma_{xx} \end{bmatrix} \begin{bmatrix} (x-\mu_x) \\ (y-\mu \ ) \end{bmatrix} / (\sigma_{xx}\sigma \quad - \sigma_x^2 \ )$$

$$= \frac{\sigma \quad (x-\mu_x)^2 - 2\sigma_x \ (x-\mu_x)(y-\mu \ ) + \sigma_{xx}(y-\mu \ )^2}{\sigma_{xx}\sigma \quad - \sigma_x^2}$$

This in turn corresponds to the generalization of the Gauss Function in 2-D:

$$G(x,y) = e^{-\frac{1}{2}\left[\sigma_{yy}(x-\mu_x)^2 - 2\sigma_{xy}(x-\mu_x)(\ -\mu_y) + \sigma_{xx}(\ -\mu_y)^2\right]/(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)}$$

A simplified quadratic form for the inverse problem can be found easily, because the eigenvalues of an inverse matrix are always the inverses of the eigenvalues of the original tensor. The latter are λ_1 and λ_2. Hence the former are found immediately to be:

$$1/\lambda_1 \quad \text{and} \quad 1/\lambda_2$$

This in turn means that the Gauss function, when transformed to eigenvector coordinates, is simply given by:

$$G(x, y) = e^{-\frac{1}{2}\left[(x-\mu_x)^2/\lambda_1+(\quad-\mu_y)^2/\lambda_2\right]}$$

What's still missing is a *norming factor* for the skewed 2-D Gaussian function. To this end, integrate the function $G(x, y)$ over the whole plane:

$$\iint G(x, y)\, dx\, dy = \iint e^{-\quad 1}$$

$R_k = \Delta_k/2$ is half the width of a discretization interval, in a further restriction on the spread σ_k:

$$\sigma_k \geq \alpha\, R_k \qquad \text{where} \quad \alpha = \frac{\sqrt{2\ln(2/\epsilon)}}{\pi}$$

In two dimensions, take a non-constant discrete function f_i, defined at the vertices (x_i, y_i) of triangles in a Finite Element mesh. Let the midpoints of the triangles (k) be $(\overline{x}_k, \overline{y}_k)$ and corresponding function values $\overline{f}_k = (f_0+f_1+f_2)/3$. Replace the lengths Δ_k by triangle areas $J_k/2$. Then:

$$\overline{f}(x,y) = \sum_k G_k(x,y)\,\overline{f}_k\, J_k/2$$

Where the distributions G are the following, now rather obvious, generalizations:

$$G(x,y) = \frac{e^{-\frac{1}{2}\left[\sigma_{yy}(x-\mu_x)^2 - 2\sigma_{xy}(x-\mu_x)(\;\;-\mu_y)+\sigma_{xx}(\;\;-\mu_y)^2\right]/(\sigma_{xx}\sigma_{yy}-\sigma_{xy}^2)}}{2\pi\sqrt{\sigma_{xx}\sigma\quad - \sigma_x^2}}$$

Here μ and σ are the first and second order moments of the triangle (k). And this is the equation of the Circumellipse, multiplied by the factor α, squared because of the squares:

$$\frac{\sigma\quad(x-\mu_x)^2 - 2\sigma_x\;\;(x-\mu_x)(y-\mu\;\;) + \sigma_{xx}(y-\mu\;\;)^2}{\sigma_{xx}\sigma\quad - \sigma_x^2} = 8\alpha^2$$

Where $\mu_x = \overline{x}$ and $\mu\;\; = \overline{y}$ and:

$$\alpha = \frac{\sqrt{2\ln(2/\epsilon)}}{\pi}$$

The denominator can be analyzed further with results from *Triangle Moments* in the prerequisite reading *Steiner Ellipses and Variances*:

$$2\pi\sqrt{\sigma_{xx}\sigma\quad - \sigma_x^2} = 2\pi\frac{\sqrt{s_{xx}s\quad - s_x^2}}{36} = \frac{2\pi\sqrt{3\,J^2}}{36} = \frac{2\pi\sqrt{3}}{36}\,J$$

And this outcome must be multiplied with $8\alpha^2$ for an outer ellipse that extends beyond the restriction on the spreads:

$$\frac{36}{2\pi\sqrt{3}\;J\;8\alpha^2} = \frac{3\sqrt{3}}{8\pi\alpha^2\;J/2}$$

Here $4\pi/(3\sqrt{3})$ is recognized as the (constant) proportion between the area of the Steiner circumellipses and the area of the circumscribed triangles. Gathering everything together, here comes the final formula. It is noted that the triangle areas $J/2$ cancel out.

$$\overline{f}(x,y) = \frac{3\sqrt{3}}{8\pi\alpha^2}\sum_\Delta G_\Delta(x,y)\,\overline{f}_\Delta$$

Where:

$$G_\Delta(x,y) = e^{-\frac{1}{2}\left[\sigma_{yy}(x-\mu_x)^2 - 2\sigma_{xy}(x-\mu_x)(\ \ -\mu_y) + \sigma_{xx}(\ \ -\mu_y)^2\right]/(8\alpha^2(\sigma_{xx}\sigma_{yy}-\sigma_{xy}^2))}$$

All the other quantites have been previously defined. There is one issue left, though: how many triangles have to be taken into account, in order to acquire a sensible approximation for the function f? To be precise, given a tolerance ϵ:

$$G_\Delta(x,y) \le \frac{1}{2}\epsilon \quad \Longleftrightarrow$$

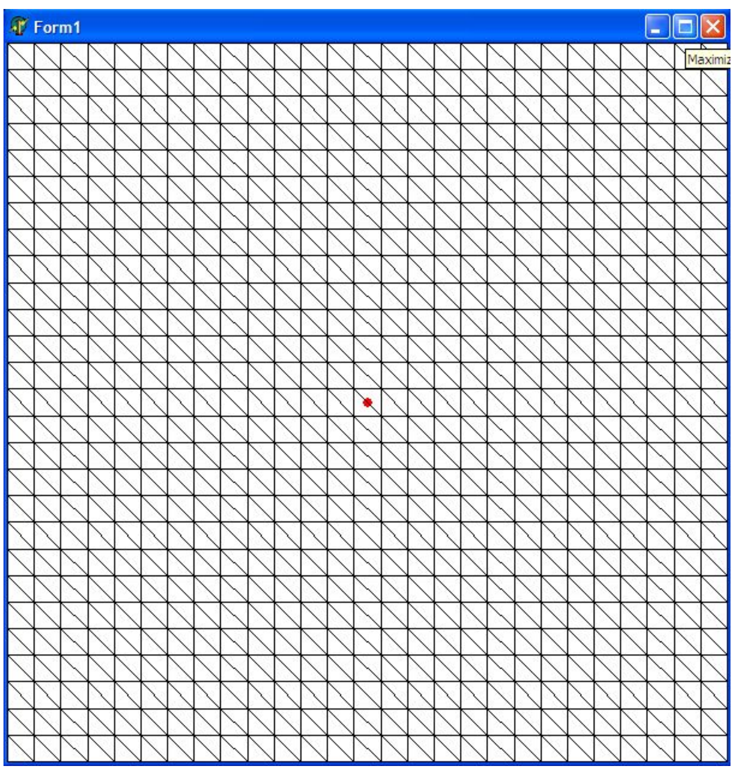
$$\frac{\sigma \quad (x-\mu_x)^2 - 2\sigma_x \ \ (x-\mu_x)(y-\mu \ \ ) + \sigma_{xx}(y-\mu \ \ )^2}{8\alpha^2(\sigma_{xx}\sigma \quad - \sigma_x^2 \ \ )} \ge 2\ln(2/\epsilon) = \pi^2\,\alpha^2$$

$$\Longleftrightarrow \quad \text{Circumellipse kernel} \ge (\pi\,\alpha^2)^2$$

Meaning that all triangles inside an ellipse of $(\pi\,\alpha^2)^2$ times the area of a Steiner circumellipse are needed. Meaning that half the size of the mesh around a point where the function value is to be evaluated must be of order $(\pi\,\alpha^2)$. For an error $\epsilon = 10^{-9}$ the numerical values are:

$$\alpha = \frac{\sqrt{2\ln(2/\epsilon)}}{\pi} \approx 2.08323609622477 \quad \text{and} \quad \pi\,\alpha^2 \approx 13.6341119801350$$

Therefore we choose the layout of the finite element mesh employed as follows.



That is: 27×27 rectangles with two triangles for each rectangle.
A patch test confirms that the theory works in practice. Unzip and run 'Project5.exe' (then search Any key and press it) and convince yourself:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2011/steiners.zip`

The patch test confirms that a Gauss-Steiner Continuization is found indeed for the constant function $f(x, y) = 1$:

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Outcome must be 1 =  9.99999999999998E-0001
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Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch :-)