# Self Energy Theory

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Observing a function in physics can be modelled by a convolution integral of this function with e.g. a rectangular distribution. Since basically everything in physics is subject to observation, the very *nature* of any function in physics is revealed by such a convolution integral. The aim of this paper is to demonstrate that a singularity of the form $1/r^2$ in three-dimensional space, such as with the Coulomb field of an electon, is actually non-existent in nature.
Herewith it is assumed that the Electron is an observer of it Self. And, as a first approximation, the Self of an electron is modelled as a little green solid sphere. Epecially mind the green :-) Serious. We certainly would have preferred a smooth transition from the electron's inner structure towards outer space, such as in "Renormalization of Singularities":

`http://hdebruijn.soo.dto.tudelft.nl/QED/singular.pdf`

But Gaussian distributions can only be done numerically, while the solid sphere approximation makes an analytical ("exact") treatment possible.

## Little Solid Sphere Everywhere

Let $R$ be the radius of the little solid sphere, $(x, y, z)$ be spatial (Cartesian) coordinates and $r, \theta, \phi$ the spherical coordinates equivalent of the latter. Then the itSelf $S$ of the electron is modelled as:

$$S(x,y,z) = \left\{ \begin{array}{lll} N & \text{for} & x^2+y^2+z^2 < R^2 \\ 0 & \text{for} & x^2+y^2+z^2 > R^2 \end{array} \right. \quad \text{or} \quad S(r) = \left\{ \begin{array}{lll} N & \text{for} & r < R \\ 0 & \text{for} & r > R \end{array} \right.$$

When seen from a great distance, the electron looks like a point charge, i.e. (ipse est): a delta function. Meaning that:

$$\lim_{R \to 0} S(x,y,z) = \delta(x,y,z)$$

Which is the reason that Self must be normed:

$$\iiint S(x,y,z)\,dx\,dy\,dz = 1 \quad \Longleftrightarrow \quad \int_0^R N\,4\pi r^2\,dr = N\,\frac{4}{3}\pi R^3 = 1 \quad \Longleftrightarrow$$

$$N = 1/\left(\frac{4}{3}\pi R^3\right)$$

The electric field $E$ of the electron is given by Coulomb's law:

$$E(r) = \frac{q}{4\pi\epsilon_0\,r^2} = \frac{q}{4\pi\epsilon_0\,(x^2+y^2+z^2)}$$

Where $q$ is the electron's charge and $\epsilon_0$ is the dielectric constant of the vacuum. By hypothesis, the electron is *everywhere*. It appears to be localized, though, due to the Coulomb field; this will be demonstrated in the sequel. The self electric field of the electron is the convolution of Self and the common Coulomb field:

$$\overline{E}(x,y,z) = \frac{q}{4\pi\epsilon_0}\iiint S(\xi,\eta,\zeta)\frac{d\xi\, d\eta\, d\zeta}{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$$

Introducing spherical coordinates:

$$\left\{\begin{array}{l} \xi = \rho\sin(\theta)\cos(\phi) \\ \eta = \rho\sin(\theta)\sin(\phi) \\ \zeta = \rho\cos(\theta) \end{array}\right. \qquad \text{where} \quad 0 \le \theta \le \pi \quad \text{and} \quad 0 \le \phi \le 2\pi$$

Because of the expected spherical symmetry of the problem, we only consider a ray in $z$ direction, which means that $x = 0$ , $y = 0$ and $z = r$ . The integral then becomes, after some suitable rearrangement:

$$\overline{E}(r) = \frac{q}{4\pi\epsilon_0}\int_0^{2\pi} d\phi \int_0^R d\rho \int_0^\pi \frac{\rho^2\sin(\theta)\, d\theta}{\rho^2\sin^2(\theta)+[r-\rho\cos(\theta)]^2} \;/\; \left(\frac{4}{3}\pi R^3\right)$$

$$= \frac{q}{4\pi\epsilon_0}\frac{3}{4\pi R^3}\,2\pi\int_0^R d\rho\int_0^\pi \frac{\rho^2\sin(\theta)\, d\theta}{\rho^2+r^2-2\,r\rho\cos(\theta)}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{3}{4\pi R^3}\,2\pi\int_0^R d\rho\frac{\rho}{2r}\int_{\theta=0}^{\theta=\pi}\frac{d\left[\rho^2+r^2-2\,r\,\rho\cos(\theta)\right]}{\rho^2+r^2-2\,r\rho\cos(\theta)}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{3}{4\pi R^3}\frac{2\pi}{2\,r}\int_0^R \rho\, d\rho\left[\ln\left(\rho^2+r^2-2\,r\,\rho\cos(\theta)\right)\right]_{\theta=0}^{\theta=\pi}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{3}{4\pi R^3}\frac{2\pi}{2\,r}\int_0^R \left\{\ln\left[(\rho+r)^2\right]-\ln\left[(\rho-r)^2\right]\right\}\rho\, d\rho$$

Substitution of $x = \rho/R$ and $s = r/R$ gives:

$$\overline{E}(r) = \frac{q}{4\pi\epsilon_0}\frac{3}{4Rr}\int_0^1 \left\{\ln\left[(x+s)^2\right]-\ln\left[(x-s)^2\right]\right\} x\, dx$$

Divide and conquer one:

$$\int \ln(x+s)^2 x\, dx =$$

$$\int \ln(x+s)^2(x+s)\, d(x+s) - s\int 2\,\ln|x+s|\, d(x+s) =$$

$$\frac{1}{2}(x+s)^2\ln(x+s)^2 - \frac{1}{2}(x+s)^2 - 2\,s\left[(x+s)\ln|x+s|-(x+s)\right] =$$

$$(x^2+2xs+s^2)\ln|x+s| - x^2/2 - xs - s^2/2$$

$$+(-2xs-2s^2)\ln|x+s| + 2xs + 2s^2 =$$

$$(x^2 - s^2)\ln|x+s| - x^2/2 + xs + 3s^2/2$$

Divide and conquer two:

$$\int \ln(x-s)^2 x\, dx =$$

$$\int \ln(x-s)^2 (x-s)\, d(x-s) + s \int 2\, \ln|x-s|\, d(x-s) =$$

$$\frac{1}{2}(x-s)^2 \ln(x-s)^2 - \frac{1}{2}(x-s)^2 + 2\, s\, [(x-s)\ln|x-s| - (x-s)] =$$

$$(x^2 - 2xs + s^2)\ln|x-s| - x^2/2 + xs - s^2/2$$

$$+(2xs - 2s^2)\ln|x-s| - 2xs + 2s^2 =$$

$$(x^2 - s^2)\ln|x-s| - x^2/2 - xs + 3s^2/2$$

One and two must be substracted. And integrated from 0 to 1.

$$\int_0^1 \left[\ln(x+s)^2 - \ln(x-s)^2\right] x\, dx =$$

$$\left[(x^2 - s^2)\ln|x+s| - x^2/2 + xs + 3s^2/2\right]_{x=0}^{x=1} -$$

$$\left[(x^2 - s^2)\ln|x-s| - x^2/2 - xs + 3s^2/2\right]_{x=0}^{x=1} =$$

$$\left[(x^2 - s^2)\ln\left|\frac{x+s}{x-s}\right| + 2\, xs\right]_{x=0}^{x=1} = (1-s^2)\ln\left|\frac{1+s}{1-s}\right| + 2\, s$$

Consequently, the Self Coulomb field of an electron is:

$$\overline{E}(r) = \frac{q}{4\pi\epsilon_0}\frac{3}{4Rr}\left\{\left[1 - \left(\frac{r}{R}\right)^2\right]\ln\left|\frac{1+r/R}{1-r/R}\right| + 2\,\frac{r}{R}\right\}$$

There are a few singularities involved here. The first one is at $r = 0$ , which is easiest to calculate if we start from the basics all over.

$$\overline{E}(0,0,0) = \frac{q}{4\pi\epsilon_0}\iiint S(\xi,\eta,\zeta)\frac{d\xi\, d\eta\, d\zeta}{\xi^2 + \eta^2 + \zeta^2} \quad \Longleftrightarrow$$

$$\overline{E}(0) = \frac{q}{4\pi\epsilon_0}\int_0^R \frac{4\pi\rho^2\, d\rho}{\rho^2} \,/\, \left(\frac{4}{3}\pi R^3\right) = \frac{q}{4\pi\epsilon_0}4\pi\, R \,/\, \left(\frac{4}{3}\pi R^3\right) = \frac{3\, q}{4\pi\epsilon_0 R^2}$$

The next singularity is where the denominator $(1 - r/R)$ in the logarithm for $r = R$ , or $(1 - s)$ for $s = 1$ , becomes zero.

$$\lim_{s\to 1}\left\{(1-s^2)\ln\left|\frac{1+s}{1-s}\right| + 2\, s\right\} =$$

$$\lim_{s\to 1}\{(1+s)(1-s)\ln|1+s| - (1+s)\,[(1-s)\ln|1-s|] + 2\, s\} = 2$$

Because $\lim_{x\to 0} x\ln(x) = 0$ with $x = (1-s)$ .

$$\Longrightarrow \quad \overline{E}(R) = \frac{q}{4\pi\epsilon_0}\frac{3}{4R.R}\,2 = \frac{3}{2}\frac{q}{4\pi\epsilon_0 R^2} = \frac{1}{2}\overline{E}(0)$$

We could have done it by hand, but with MAPLE it goes faster:

```
> simplify(diff((1-s^2)*ln((1+s)/(1-s))+2*s,s));
```

$$-2s\ln\left|\frac{1+s}{1-s}\right|+4$$

From this we see that for $s=1$ i.e. $r=R$ the function $\overline{E}(r)$ has a slope that is negative and infinitely large: $\overline{E'}(R)=-\infty$ .
The Self Coulomb field of an electron can also be written as follows:

$$\overline{E}(r)=\frac{q}{4\pi\epsilon_0}\frac{3}{4R^2(R/r)}\left\{\left[\left(\frac{R}{r}\right)^2-1\right]\ln\left|\frac{R/r+1}{R/r-1}\right|+2\,\frac{R}{r}\right\}$$

And with help of the variable $x=R/r$ , which is small for $r>>R$ :

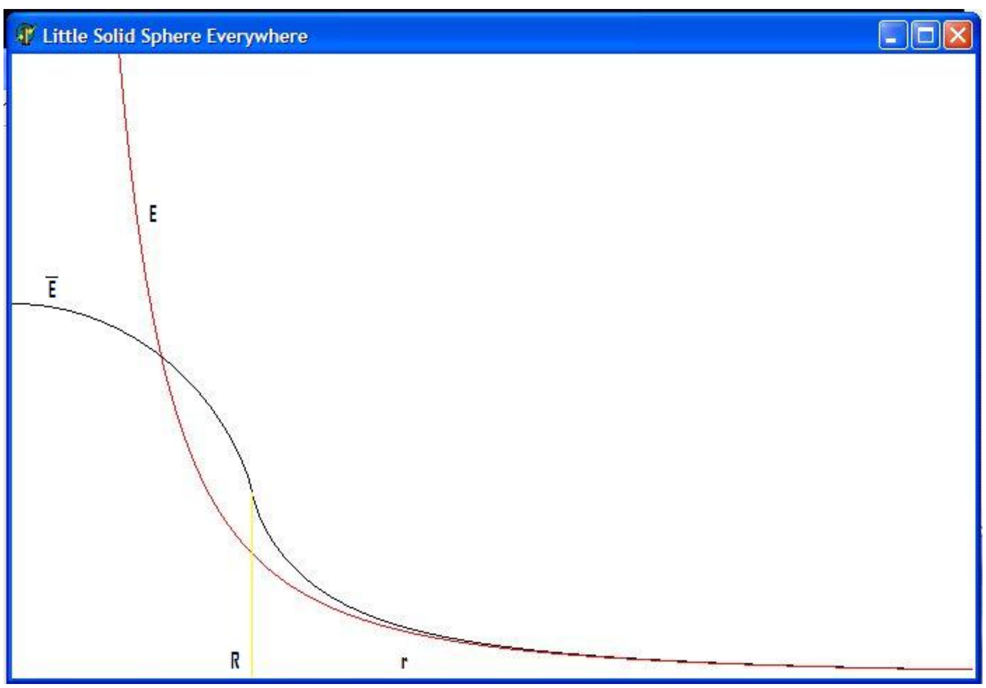
$$=\frac{q}{4\pi\epsilon_0}\frac{3}{4R^2x}\left\{(x^2-1)\ln\left|\frac{x+1}{x-1}\right|+2\,x\right\}$$

We could have done it by hand, but with MAPLE it goes much faster:

```
> series(3/4/(R^2*x)*((x^2-1)*ln((1+x)/(1-x))+2*x),x,7);
```

$$\frac{1}{R^2}x^2+\frac{1}{5R^2}x^4+\frac{3}{35R^2}x^6+O(x^8)\approx\frac{1}{R^2}\left(\frac{R}{r}\right)^2=\frac{1}{r^2}$$

Thus the standard Coulomb law (red) is valid for small values of $x$ , that is for large values of $r$ , that is for $r>>R$ . Self Coulomb law is depicted in black; yellow line at position $r=R$ . See picture:



At last, we want to calculate the Self Energy $U$ of the electron, as defined by the formula ($V$ = space volume):

$$U = \iiint \frac{1}{2}\epsilon_0 \overline{E}^2 \, dV = \int_0^\infty \frac{1}{2}\epsilon_0 \overline{E}^2(r)\, 4\pi r^2 \, dr$$

Where $x = r/R$ in:

$$\overline{E}(r) = \frac{q}{4\pi\epsilon_0}\frac{3}{4R^2 r/R}\left\{\left[1-\left(\frac{r}{R}\right)^2\right]\ln\left|\frac{1+r/R}{1-r/R}\right| + 2\,\frac{r}{R}\right\} =$$

$$\frac{q}{4\pi\epsilon_0}\frac{3}{4R^2 x}\left\{(1-x^2)\ln\left|\frac{1+x}{1-x}\right| + 2\,x\right\} \quad \Longrightarrow$$

$$U = \frac{q^2}{8\pi\epsilon_0 R}\left(\frac{3}{4}\right)^2 \int_0^\infty \left\{(1-x^2)\ln\left|\frac{1+x}{1-x}\right| + 2\,x\right\}^2 dx$$

We could *not* have done this by hand, so let's see what MAPLE says about it:

```
> f(x) := ((1-x^2)*ln(abs((1+x)/(1-x)))+2*x)^2;
> int(expand(f(x)),x=0..infinity);
```

And the outcome is truly wonderful ..

$$\frac{8\pi^2}{15} \qquad \text{YES !!}$$

Hence the Self Energy of the Electron is, according to this model:

$$U = \frac{q^2}{8\pi\epsilon_0 R}\left(\frac{3}{4}\right)^2 \frac{8\pi^2}{15} = \frac{3\,\pi^2}{10}\frac{q^2}{8\pi\epsilon_0 R}$$

Therefore the radius of our little green solid sphere is roughly three times the classical electron radius $a$ :

$$U = \frac{q^2}{8\pi\epsilon_0\, a} \quad \Longleftrightarrow \quad R = \frac{3\,\pi^2}{10}\, a$$

## Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch :-)