Nice Integral

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Dated: 2010 December

Theorem.

$$\int_0^\infty \left[\arctan\left(\frac{1}{r}\right)\right]^2 dr = \pi \ln(2)$$

Proof. Let $x = \arctan(1/r)$ then $r = 1/\tan(x)$, $(r \to 0) \iff (x \to \pi/2)$, $(r \to \infty) \iff (x \to 0)$, giving:

$$\int_0^\infty \left[\arctan\left(\frac{1}{r}\right)\right]^2 dr = \int_{\pi/2}^0 x^2 \, d\frac{1}{\tan(x)} = \left[\frac{x^2}{\tan(x)}\right]_{\pi/2}^0 + 2\int_0^{\pi/2} \frac{x\,dx}{\tan(x)} =$$

$$2\int_0^{\pi/2} \frac{x\,d\sin(x)}{\sin(x)} = 2\left[x\ln(\sin x)\right]_0^{\pi/2} - 2\int_0^{\pi/2} \ln(\sin x)\,dx = -2\int_0^{\pi/2} \ln(\sin x)\,dx$$

The latter integral is known to be equal to $-(\pi \ln 2)/2$, according to:

`http://math.ucsd.edu/~ebender/20B/7_DefInt.pdf`

Notes. There is a typo in this article: $\int \ln x dx = x\ln x + 1$ must be $\int \ln x dx = x \ln x + x$. Furthermore, it must be known that $\frac{2}{\pi}x \le \sin x \le 1$, which can be verified most easily by drawing a graph of the sine and a few lines. And it must be known that $\lim_{x\to 0} x \ln x = 0$. The latter can eventually be proved with L'Hôpital's Rule:

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0}(-x) = 0$$

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch :-)