Conic Sections

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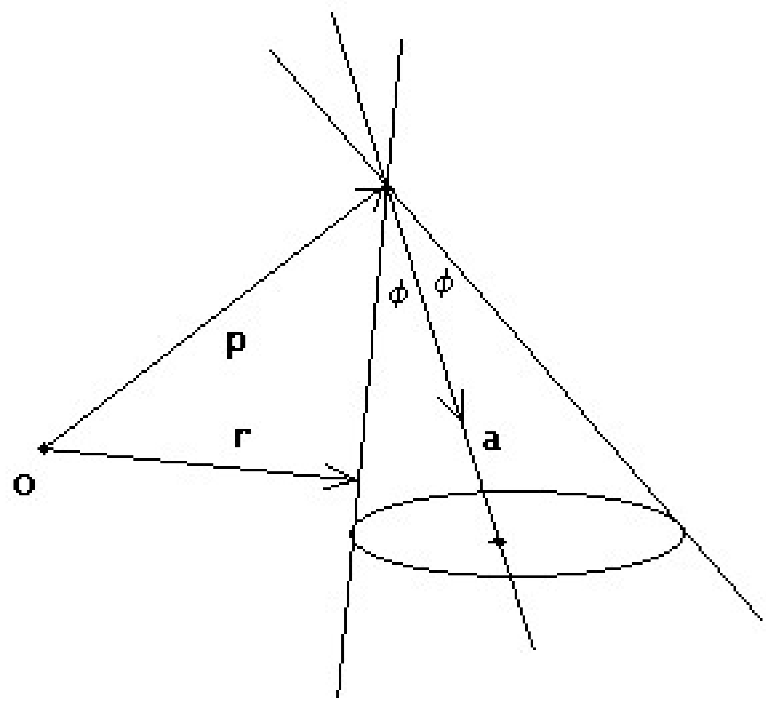
The theory of Conic Sections stems from ancient times. It is an example of pure mathematics, which has found applications only many centuries after it has been developed: with the laws of planet motion as discovered by Johannes Keppler. But, quite interesting as it is, we shall leave aside history and come to core - or rather cone - business almost immediately.



Analysis

The problem is to intersect a circular cone with a plane and determine the curves of intersection, like shown in the above figure. This could be done in the way the old Greek mathematicians did it. But we prefer to drive along a road that requires less ingenuity and we will employ the means of modern analytical geometry instead.

A circular cone is characterized by the fact that the angle ϕ between the cone axis and its surface is a constant. Let the unit vector $\vec{a}$ be the direction of the cone axis and let $\vec{p}$ point to the top vertex of the cone. An arbitrary point at the surface of the cone is pinpointed by $\vec{r}$. Then the following is an equation of the cone surface:

$$(\vec{a} \cdot \vec{r} - \vec{p}) = |\vec{a}||\vec{r} - \vec{p}| \cos(\phi)$$

Square both sides:

$$(\vec{a} \cdot \vec{r} - \vec{p})^2 = (\vec{a} \cdot \vec{a})(\vec{r} - \vec{p} \cdot \vec{r} - \vec{p}) \cos^2(\phi)$$

And work out:

$$(\vec{a} \cdot \vec{r})^2 - 2(\vec{a} \cdot \vec{p})(\vec{a} \cdot \vec{r}) + (\vec{a} \cdot \vec{p})^2 = \cos^2(\phi) \{(\vec{r} \cdot \vec{r}) - 2(\vec{p} \cdot \vec{r}) + (\vec{p} \cdot \vec{p})\}$$

The unit vector $\vec{a}$ can be written as:

$$\vec{a} = [\cos(\alpha)\cos(\gamma), \cos(\alpha)\sin(\gamma), \sin(\alpha)]$$

Where α is the angle between the cone axis and the XY-plane and γ is an angle that indicates how the conic section is rotated in the plane. The vector of the top of the cone can be written in its coordinates as:

$$\vec{p} = (p, q, h)$$

Where h is the height of the cone above the XY plane and (p, q) indicates how the conic section is translated in the plane. Last but not least, the vector pointing to the cone surface is written as:

$$\vec{r} = (x, y, z)$$

Where the intersections with the XY plane are found for $z = 0$. Let's do just that and work out the above:

$$\begin{aligned}
(\vec{a} \cdot \vec{r}) &= \cos(\alpha)\cos(\gamma)\, x + \cos(\alpha)\sin(\gamma)\, y \\
(\vec{a} \cdot \vec{p}) &= \cos(\alpha)\cos(\gamma)\, p + \cos(\alpha)\sin(\gamma)\, q + \sin(\alpha)\, h \\
(\vec{r} \cdot \vec{r}) &= x^2 + y^2 \\
(\vec{p} \cdot \vec{r}) &= p\, x + q\, y \\
(\vec{p} \cdot \vec{p}) &= p^2 + q^2 + h^2
\end{aligned}$$

Resulting in:

$$\begin{gathered}
[\cos(\alpha)\cos(\gamma)\, x + \cos(\alpha)\sin(\gamma)\, y]^2 \\
-2(\vec{a} \cdot \vec{p})\, [\cos(\alpha)\cos(\gamma)\, x + \cos(\alpha)\sin(\gamma)\, y] \\
+(\vec{a} \cdot \vec{p})^2 = \cos^2(\phi) \left\{x^2 + y^2 - 2(p\, x + q\, y) + (\vec{p} \cdot \vec{p})\right\}
\end{gathered}$$

Collecting powers of x and y results in:

$$A\, x^2 + B\, xy + C\, y^2 + D\, x + E\, y + F = 0$$

Where:

$$\begin{aligned}
A &= \cos^2(\phi) - \cos^2(\alpha)\cos^2(\gamma) \\
B &= -2\cos^2(\alpha)\cos(\gamma)\sin(\gamma) \\
C &= \cos^2(\phi) - \cos^2(\alpha)\sin^2(\gamma) \\
D &= 2\left\{\cos(\alpha)\cos(\gamma)(\vec{a}\cdot\vec{p}) - \cos^2(\phi)\,p\right\} \\
E &= 2\left\{\cos(\alpha)\sin(\gamma)(\vec{a}\cdot\vec{p}) - \cos^2(\phi)\,q\right\} \\
F &= (\vec{p}\cdot\vec{p})\cos^2(\phi) - (\vec{a}\cdot\vec{p})^2
\end{aligned}$$

Here:

$$D = 2\left\{\cos(\alpha)\cos(\gamma)\left[\cos(\alpha)\cos(\gamma)\,p + \cos(\alpha)\sin(\gamma)\,q + \sin(\alpha)\,h\right] - \cos^2(\phi)\,p\right\} =$$

$$2\left[\cos^2(\alpha)\cos^2(\gamma) - \cos^2(\phi)\right]p + 2\cos^2(\alpha)\cos(\gamma)\sin(\gamma)\,q + 2\cos(\alpha)\sin(\alpha)\cos(\gamma)\,h$$

$$\Longrightarrow \quad D = -2A\,p - B\,q + \sin(2\alpha)\cos(\gamma)\,h$$

And:

$$E = 2\left\{\cos(\alpha)\sin(\gamma)\left[\cos(\alpha)\cos(\gamma)\,p + \cos(\alpha)\sin(\gamma)\,q + \sin(\alpha)\,h\right] - \cos^2(\phi)\,q\right\} =$$

$$2\cos^2(\alpha)\cos(\gamma)\sin(\gamma)\,p + 2\left[\cos^2(\alpha)\sin^2(\gamma) - \cos^2(\phi)\right]q + 2\cos(\alpha)\sin(\alpha)\sin(\gamma)\,h$$

$$\Longrightarrow \quad E = -B\,p - 2C\,q + \sin(2\alpha)\sin(\gamma)\,h$$

Last but not least:

$$\begin{aligned}
F &= (p^2 + q^2 + h^2)\cos^2(\phi) - \left[\cos(\alpha)\cos(\gamma)\,p + \cos(\alpha)\sin(\gamma)\,q + \sin(\alpha)\,h\right]^2 \\
&= p^2\left[\cos^2(\phi) - \cos^2(\alpha)\cos^2(\gamma)\right] + q^2\left[\cos^2(\phi) - \cos^2(\alpha)\sin^2(\gamma)\right] \\
&\quad + p\left[-2\cos(\alpha)\sin(\alpha)\cos(\gamma)\ h\right] + q\left[-2\cos(\alpha)\sin(\alpha)\sin(\gamma)\ h\right] \\
&\quad + pq\left[-2\cos^2(\alpha)\cos(\gamma)\sin(\gamma)\right] + h^2\left[\cos^2(\phi) - \sin^2(\alpha)\right] \quad \Longrightarrow \\
F &= Ap^2 + Bpq + Cq^2 - h\sin(2\alpha)\cos(\gamma)p - h\sin(2\alpha)\sin(\gamma)q \\
&\quad + h^2\left[\cos^2(\phi) - \sin^2(\alpha)\right]
\end{aligned}$$

With help of the expressions for D and E:

$$\begin{aligned}
F &= Ap^2 + Bpq + Cq^2 - (D + 2Ap + Bq)p - (E + Bp + 2Cq)q \\
&\quad + h^2\left[\cos^2(\phi) - \sin^2(\alpha)\right] \quad \Longrightarrow \\
F &= h^2\left[\cos^2(\phi) - \sin^2(\alpha)\right] - (Ap^2 + Bpq + Cq^2 + Dp + Eq) \\
&\Longrightarrow \quad Ap^2 + Bpq + Cq^2 + Dp + Eq + F = h^2\left[\cos^2(\phi) - \sin^2(\alpha)\right]
\end{aligned}$$

Many other expressions can be derived. But not all of them appear to be equally interesting.

Meaning

The first three coefficients of the conic section equation are:

$$\begin{aligned} A &= \cos^2(\phi) - \cos^2(\alpha)\cos^2(\gamma) \\ B &= -2\cos^2(\alpha)\cos(\gamma)\sin(\gamma) \\ C &= \cos^2(\phi) - \cos^2(\alpha)\sin^2(\gamma) \end{aligned}$$

All kind of conics can still be produced if the angles ϕ and α are limited to sensible values:

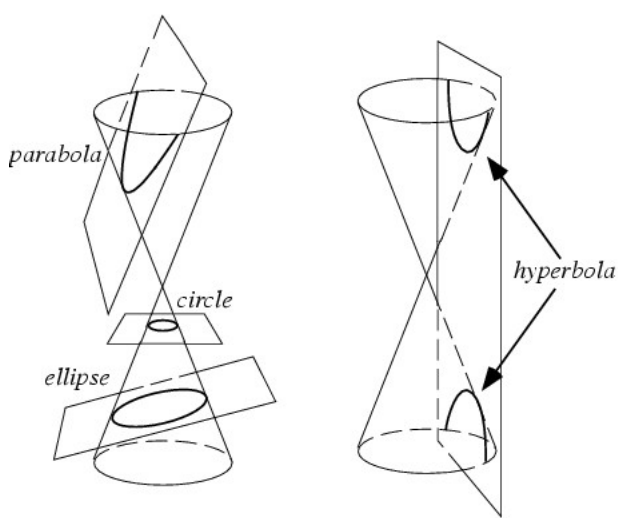
$$\begin{aligned} 0 < \phi < 90^o &\implies 0 < \cos(\phi) < 1 \\ 0 \le \alpha \le 90^o &\implies 0 \le \cos(\alpha) \le 1 \end{aligned}$$

Generality is not affected by these choices. Moreover it is seen from the sole picture in this document that the *form* of the conic section is determined by the angles ϕ and α and nothing else. Therefore the ratio of the two angles will be defined here - somewhere else ? - as the **excentricity** (ϵ) of the conic section:

$$\epsilon = \frac{\cos(\alpha)}{\cos(\phi)}$$

The following relationships exist between the excentricity and the form of a conic section, as is clear from the figure:

$$\begin{aligned} \text{Circle}: &\quad \alpha = 90^o &\iff& \quad \epsilon = 0 \\ \text{Ellipse}: &\quad \alpha > \phi &\iff& \quad \epsilon < 1 \\ \text{Parabola}: &\quad \alpha = \phi &\iff& \quad \epsilon = 1 \\ \text{Hyperbola}: &\quad \alpha < \phi &\iff& \quad \epsilon > 1 \end{aligned}$$



Where it is noted that the circle can be considered as a special case of an ellipse.

So far so good. The coefficients (A, B, C) can be combined into some interesting quantities which are only dependent upon form, that is: the angles ϕ and α. It is remarked in the first place that (A, B, C) are independent of the vector $\vec{p} = (p, q, h)$ and thus independent of translation and scaling. If we seek to eliminate any dependence upon the angle of rotation γ, then we find:

$$A + C = 2\cos^2(\phi) - \cos^2(\alpha) = \cos^2(\phi)(2 - \epsilon^2)$$

This quantity is known (for some good reasons) as the **trace** of the conic section. Instead of eliminating the angle of rotation, we could also try to calculate it.

$$A - C = -\cos^2(\alpha)\left[\cos^2(\gamma) - \sin^2(\gamma)\right] = -\cos^2(\alpha)\cos(2\gamma)$$

There is a striking resemblance with:

$$B = -\cos^2(\alpha)\sin(2\gamma)$$

We thus find:

$$\frac{B}{A - C} = \frac{\sin(2\gamma)}{\cos(2\gamma)} \quad \Longrightarrow \quad \tan 2\gamma = \frac{B}{A - C}$$

Herewith - in principle - the angle of rotation γ can be reconstructed from the conic section equation; provided that $A \neq C$.
Let's proceed with another quantity that is independent of any rotation.

$$B^2 - 4AC = \left[-2\cos^2(\alpha)\cos(\gamma)\sin(\gamma)\right]^2$$

$$-4\left[\cos^2(\phi) - \cos^2(\alpha)\cos^2(\gamma)\right]\left[\cos^2(\phi) - \cos^2(\alpha)\sin^2(\gamma)\right]$$

This quantity is known (also for some good reasons) as the **determinant** or **discriminant** of the conic section. Work out:

$$= 4\cos^4(\alpha)\cos^2(\gamma)\sin^2(\gamma) - 4\cos^4(\phi) - 4\cos^4(\alpha)\cos^2(\gamma)\sin^2(\gamma)$$

$$+4\cos^2(\phi)\cos^2(\alpha)\left[\cos^2(\gamma) + \sin^2(\gamma)\right]$$

$$= -4\cos^4(\phi) + 4\cos^2(\phi)\cos^2(\alpha)$$

$$\Longrightarrow \quad B^2 - 4AC = 4\cos^2(\phi)\left[\cos^2(\alpha) - \cos^2(\phi)\right] = \left[2\cos^2(\phi)\right]^2(\epsilon^2 - 1)$$

The following relationships exist between the discriminant and the form of a conic section, as is clear from the above:

$$\begin{array}{ccccc} \text{Ellipse} & \Longleftrightarrow & \epsilon < 1 & \Longleftrightarrow & (B^2 - 4AC) < 0 \\ \text{Parabola} & \Longleftrightarrow & \epsilon = 1 & \Longleftrightarrow & (B^2 - 4AC) = 0 \\ \text{Hyperbola} & \Longleftrightarrow & \epsilon > 1 & \Longleftrightarrow & (B^2 - 4AC) > 0 \end{array}$$

So far so good. On the other hand:

$$(A + C)^2 = 4\cos^2(\phi)\left[\cos^2(\phi) - \cos^2(\alpha)\right] + \cos^4(\alpha)$$

Upon addition this gives:

$$(B^2 - 4AC) + (A+C)^2 = \cos^4(\alpha) \quad \Longrightarrow$$

$$\cos(\alpha) = \sqrt{\sqrt{B^2 + (A-C)^2}}$$

About the angle ϕ between the cone axis and its surface:

$$A + C = 2\cos^2(\phi) - \sqrt{B^2 + (A-C)^2} \quad \Longrightarrow$$

$$\cos(\phi) = \sqrt{\frac{(A+C) + \sqrt{B^2 + (A-C)^2}}{2}}$$

Herewith the excentricity ϵ can be expressed into the coefficients of the conic section equation (A, B, C):

$$\epsilon = \sqrt{\frac{2\sqrt{B^2 + (A-C)^2}}{(A+C) + \sqrt{B^2 + (A-C)^2}}}$$

Specialization

There is a special case which must be studied before the other special cases can be studied. It is the conic section called **parabola**. A parabola is characterized by the fact that its excentricity is one, which is equivalent to the fact that its discriminant is zero:

$$\epsilon = 1 \quad \Longleftrightarrow \quad (B^2 - 4AC) = 0 \quad \Longleftrightarrow \quad B = \pm 2\sqrt{AC}$$

The general equation of a conic section has been found to be:

$$A\,x^2 + B\,xy + C\,y^2 + D\,x + E\,y + F = 0$$

Upon substitution this becomes the general equation of a parabola:

$$A\,x^2 \pm 2\sqrt{AC}\,xy + C\,y^2 + D\,x + E\,y + F = 0$$

$$\Longleftrightarrow \quad (\sqrt{A}\,x \pm \sqrt{C}\,y)^2 + D\,x + E\,y + F = 0$$

Where it follows from $\epsilon = 1$ that $\cos(\phi) = \cos(\alpha)$. Giving:

$$A = \cos^2(\phi) - \cos^2(\alpha)\cos^2(\gamma) = \cos^2(\phi)\left[1 - \cos^2(\gamma)\right] = \cos^2(\phi)\sin^2(\gamma)$$

$$\Longrightarrow \quad \sqrt{A} = \cos(\phi)\sin(\gamma)$$

$$C = \cos^2(\phi) - \cos^2(\alpha)\sin^2(\gamma) = \cos^2(\phi)\left[1 - \sin^2(\gamma)\right] = \cos^2(\phi)\cos^2(\gamma)$$

$$\Longrightarrow \quad \sqrt{C} = \cos(\phi)\cos(\gamma)$$

Special choices for γ are $\gamma = 0^o$ or $\gamma = 90^o$. Resulting in $\sqrt{A} = 0$ or $\sqrt{C} = 0$ respectively. The second case leads us to the most common (standard) equation of the parabola. Assuming that $E \neq 0$ and $a = A/E, b = D/E, c = F/E$:

$$A\,x^2 + D\,x + E\,y + F = 0 \quad \text{or} \quad y = a\,x^2 + b\,x + c$$

General expressions for D and E are:

$$\begin{aligned} D &= -2A\,p - B\,q + \sin(2\alpha)\cos(\gamma)\,h \\ E &= -B\,p - 2C\,q + \sin(2\alpha)\sin(\gamma)\,h \end{aligned}$$

Most of the time, it is possible to reformulate the conic section equation as a so-called midpoint equation, that is: in a coordinate system where $D = E = 0$. In order to accomplish this, we must solve the following equations for (p, q):

$$\begin{aligned} 2A\,p + B\,q &= \sin(2\alpha)\cos(\gamma)\,h \\ B\,p + 2C\,q &= \sin(2\alpha)\sin(\gamma)\,h \end{aligned}$$

Two linear equations with two unknowns. This small system has a solution if its determinant $(B^2 - 4AC)$ is just non-zero. But hey ! Now we understand where the word "determinant" comes from. And it's also clear now why the parabolic case is a logical first choice. Because, apparently, a midpoint equation cannot be found if the conic section is a parabola. But for all other cases, it should work. Thus we have, for $B^2 - 4AC \neq 0$, the following general midpoint equation for a conic section:

$$A\,x^2 + B\,xy + C\,y^2 + F = 0$$

Before proceeding, we have discovered that, apart from the determinant, there exists another rotation angle independent quantity. It is the **trace**:

$$A + C = 2\cos^2(\phi) - \cos^2(\alpha) = \cos^2(\phi)(2 - \epsilon^2)$$

The trace is apparently zero for $\epsilon = \sqrt{2}$, hence for $\cos^2(\alpha) = 2\cos^2(\phi)$. So this must be a kind of hyperbola, with main coefficients:

$$\begin{aligned} A &= \cos^2(\phi) - 2\cos^2(\phi)\cos^2(\gamma) = -\cos^2(\phi)\cos(2\gamma) \\ B &= -4\cos^2(\phi)\cos(\gamma)\sin(\gamma) = -2\cos^2(\phi)\sin(2\gamma) \\ C &= \cos^2(\phi) - 2\cos^2(\phi)\sin^2(\gamma) = +\cos^2(\phi)\cos(2\gamma) \end{aligned}$$

The coefficients A and C are zero for $\gamma = 45^o$ while the coefficient B is zero for $\gamma = 0^o$ or $\gamma = 90^o$. Thus the midpoint equation for that hyperbola assumes two possible special forms:

$$\begin{aligned} B\,xy + F = 0 \quad &\Longrightarrow \quad xy = c \quad \text{where} \quad c = F/B \\ A\,x^2 - A\,y^2 + F = 0 \quad &\Longrightarrow \quad x^2 - y^2 = c \quad \text{where} \quad c = F/A \end{aligned}$$

In both cases, the result is known as an **othogonal hyperbola**.
Let's proceed now with the rest of the conic sections, known as hyperbolas and ellipses. Their midpoint equation is, again:

$$A\,x^2 + B\,xy + C\,y^2 + F = 0$$

This equation can be simplified further by assuming that the rotation angle is either $\gamma = 0^o$ or $\gamma = 90^o$ in:

$$B = -2\cos^2(\alpha)\cos(\gamma)\sin(\gamma) = 0$$

Resulting in the so-called (almost) standard form for ellipses and hyperbolas:

$$A\,x^2 + C\,y^2 + F = 0$$

Where, for **ellipses**:

$$B^2 - 4AC < 0 \quad \Longrightarrow \quad (A > 0 \quad \text{and} \quad C > 0) \quad \text{or} \quad (A > 0 \quad \text{and} \quad C > 0)$$

And, for **hyperbolas**:

$$B^2 - 4AC > 0 \quad \Longrightarrow \quad (A > 0 \quad \text{and} \quad C < 0) \quad \text{or} \quad (A < 0 \quad \text{and} \quad C > 0)$$

For ellipses, if we put $A/F = -1/a^2$ and $C/F = -1/b^2$ a very much standard equation is the result:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

We are curious of what has become of our excentricity, for ellipses only:

$$\epsilon = \sqrt{\frac{2\sqrt{B^2 + (A-C)^2}}{(A+C) + \sqrt{B^2 + (A-C)^2}}}$$

Assuming that $a > b$, it follows that:

$$\epsilon = \sqrt{\frac{2(1/b^2 - 1/a^2)}{(1/a^2 + 1/b^2) + (1/b^2 - 1/a^2)}} \quad \Longrightarrow \quad \epsilon = \sqrt{1 - \frac{b^2}{a^2}}$$

Which is, indeed, much more like a well-known expression for the excentricity of an ellipse.

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch.