Special Theory of Continuity

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After having finished the *Uniform Combs of aussians* project, uniform combs of other *hat functions* will be investigated. We start with a general approach, followed by kind of heuristics. Then we specialize for respectively the Cauchy distribution, triangle, rectangle, exponential decay, sinc function and other band limited hat functions. The latter will lead us, through the Nyquist - Shannon sampling theorem, towards the more *eneral Theory of Continuity*. Later.

Uniform Combs of Hat functions

The subject of our current study are uniform combs $P(x)$ of normed *hat* shaped curves / functions $p(x)$. Such combs are defined as:

$$P(x) = \sum_{L=-\infty}^{+\infty} p(x - L.\Delta)\,\Delta$$

Where Δ is the discretization interval length and where being *normed* means that:

$$\int_{-\infty}^{+\infty} p(x)\,dx = 1$$

In addition, all hat functions are assumed to be *symmetrical* around $x = 0$:

$$p(-x) = p(x)$$

Given a sufficient refinement of the discretization Δ - to be defined later - the comb $P(x)$ can be interpreted as a Riemann sum, approximating the following integral, with $\Delta \to d\xi$. This explains the factor Δ in the above definition.

$$\lim_{\Delta \to 0} P(x) = \int_{-\infty}^{+\infty} p(x - \xi)\,d\xi = 1$$

The function $P(x)$ can be interpreted as an attempt to "smooth" the uniform density $x_L = L.\Delta$. Or "make fuzzy" the discretization $f(x_L) = 1$ of a constant - and continuous - function $f(x) = 1$. This could be called the Special Theory of Continuity.
It is easily shown that the above function $P(x)$ is *periodic*. Its period is equal to Δ: $P(x + \Delta) = P(x)$ for arbitrary x. Meaning that $P(x)$ can be developed into a *Fourier series*. The Fourier series of any periodic function is given by:

$$P(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k\, cos(k\omega x)$$

But, in addition, the function is *even*, meaning that $P(x) = P(-x)$, which results in real-valued Fourier coefficients a_k :

$$a_k = \frac{1}{\Delta/2} \int_{-\Delta/2}^{+\Delta/2} P(x) \cos(k\, 2\pi/\Delta\, x)\, dx$$

In the sequel, kind of an angular frequency ω will stand for the quantity $= 2\pi/\Delta$. Then let the calculations continue:

$$= \frac{1}{\Delta/2} \int_{-\Delta/2}^{+\Delta/2} \sum_{L=-\infty}^{+\infty} p(x - L\Delta)\, \Delta \cos(k\omega x)\, dx$$

$$= 2 \times \sum_{L=-\infty}^{+\infty} \int_{-\Delta/2}^{+\Delta/2} p(x - L\Delta) \cos(k\omega x)\, dx$$

Substitute $y = x - L\Delta$ and integrate to y:

$$a_k/2 = \sum_{L=-\infty}^{+\infty} \int_{-\Delta/2-L\Delta}^{+\Delta/2-L\Delta} p(y) \cos(k\omega[y + L\Delta])\, dy$$

Where:

$$\cos(k\omega[y + L\Delta]) = \cos(k\omega y + k.L.2\pi) = \cos(k\omega y)$$

Next replace y by $-y$ and switch integration bounds:

$$a_k/2 = \sum_{L=-\infty}^{+\infty} \int_{L\Delta-\Delta/2}^{L\Delta+\Delta/2} p(y) \cos(k\omega y)\, dy$$

The above integrals are precisely the adjacent pieces of another integral which has bounds reaching to infinity. That is, they sum up to an infinite integral:

$$a_k/2 = \int_{-\infty}^{+\infty} p(y) \cos(k\omega y)\, dy$$

Now the (continuous) Fourier integral of $p(x)$ is defined by:

$$A(y) = \int_{-\infty}^{+\infty} p(x) \cos(xy)\, dx$$

Wherefrom it is concluded that the (discrete) coefficients of the Fourier series are a *sampling* of the (continuous) Fourier integral:

$$a_k/2 = A(k\omega)$$

And especially:

$$a_0/2 = A(0) = \int_{-\infty}^{+\infty} p(x)\, dx \quad \Longrightarrow \quad \frac{1}{2} a_0 = 1$$

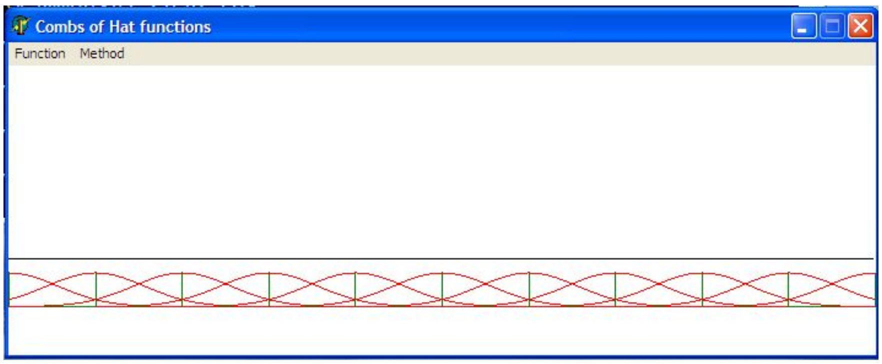
Therefore the general expression for the Fourier series of a uniform comb of hat functions is:

$$P(x) = 1 + 2 \times \sum_{k=1}^{\infty} A(k\omega)\, cos(k\omega x)$$

Where it is reminded that $\omega = 2\pi/\Delta$. And:

$$A(y) = \int_{-\infty}^{+\infty} p(x)\cos(xy)\, dx$$

It is seen herefrom that $P(x)$, indeed, is an approximation of the constant function $f(x) = 1$, provided that the rest of the Fourier series is just a minor correction on this value. At hand of a few sample hat functions $p(x)$, we will investigate if such is the rule or rather an exception.



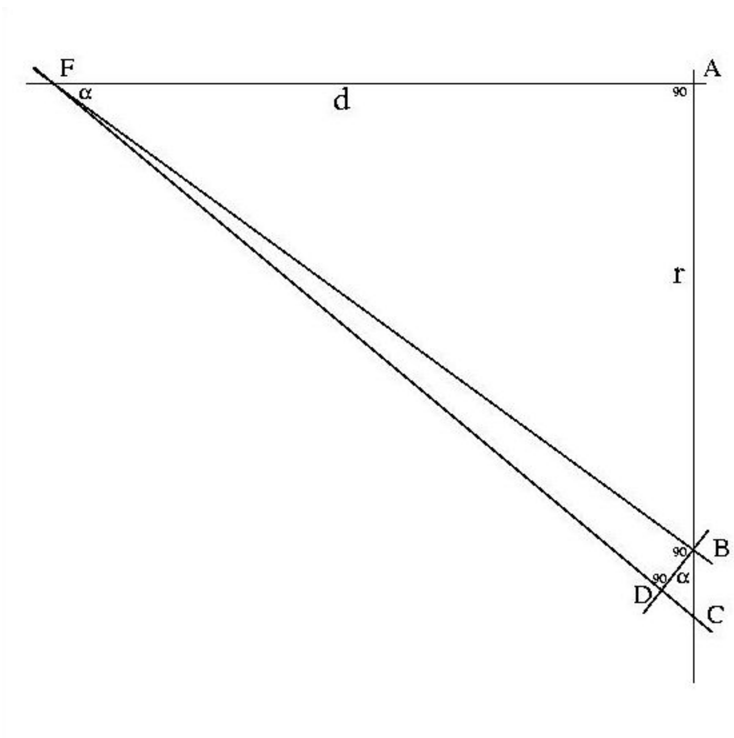
`http://hdebruijn.soo.dto.tudelft.nl/jaar2010/dikte/document.pdf`

In the article *Uniform Combs of aussians* we have already encountered an outstanding example of *Uniform Combs of Hat functions.* When cast in the standard form, as defined above, it reads:

$$\sum_{L=-\infty}^{+\infty} \frac{\Delta}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(x-L\Delta)^2/\sigma^2} = 1 + 2 \times \sum_{k=1}^{\infty} e^{-\frac{1}{2}(k\omega\sigma)^2}\, cos(k\omega x)$$

Fuzzy Optics

Sometimes you can see things better with your eyes half shut. Maybe there is no more lucid way than this for expressing the idea of *continuity.* In the physics lessons at school, a little piece of geometric optics has always been part of the program: convex and concave mirrors and lenses. Herewith it is assumed, quite naturally, that any image of an object is *crisp* and clear. My proposal here is to say goodbye to this good habit, and pay attention to *fuzzy* images instead. In the figure below, much enlarged, we see the geometry of such a fuzzy image:



Usually, a crisp image is formed at the spot F. However, now suppose that the image plane is shifted a little bit to the right, over a distance d . Consider a very narrow light bundle F C. The bundle fans out slowly and hits the image plane at C. Because the bundle is very narrow, both the angles F D and FD are approximately 90 degrees. This means that the angle C D will be approximately equal to the angle FA. Name this angle α. The light density P at the surface C shall be calculated. Assume that the light is emitted by a point source with strength 1, then: $P = cos(\alpha)/(2\pi R^2)$ (: half sphere). Here $cos(\alpha) = d/R$ and $R = \sqrt{r^2 + d^2}$. If the surface C is contracted to a point, then we find for the light strength in place the following "exact" expression:

$$P(r) = \frac{d/(2\pi)}{(r^2 + d^2)^{3/2}}$$

It is noted that the derivation with help of the approximately straight angles F D and FD is motivated only afterwards, as the limit "has been taken". This is a typical example of a "derivation with pain", as it is applied quite frequently in the applied sciences / physics.
A few things are noted. At first that a crisp image is obtained (a *delta function* to be precise) as soon as the distance d approaches zero. Integration of the formula over the whole image plane obviously must yield a total amount of light

equal to 1 . This can be checked out:

$$\iint \frac{d/2\pi}{(r^2+d^2)^{3/2}}\, r.dr.d\phi = 2\pi.\frac{1}{2\pi}.\int_0^{\infty} \frac{r/d.d(r/d)}{[(r/d)^2+1]^{3/2}} = -\frac{2}{2}.\left[x^{-1/2}\right]_1^{\infty} = 1$$

If a straight line is considered, instead of a point-like light source, then the function P must be integrated all over this line. Along the line, measurement is defined with a length l. The radius r in the above formulas is replaced by $p^2 + l^2$, where p is the distance of the point (x, y) to the line. The integration procedure therefore is as follows:

$$L = \int_{-\infty}^{+\infty} \frac{d/2\pi}{(p^2+l^2+d^2)^{3/2}}\, dl = \frac{d/2\pi}{p^2+d^2} \int_{-\infty}^{+\infty} \frac{d\left(\frac{l}{\sqrt{p^2+d^2}}\right)}{\left[1+\left(\frac{l}{\sqrt{p^2+d^2}}\right)^2\right]^{3/2}}$$

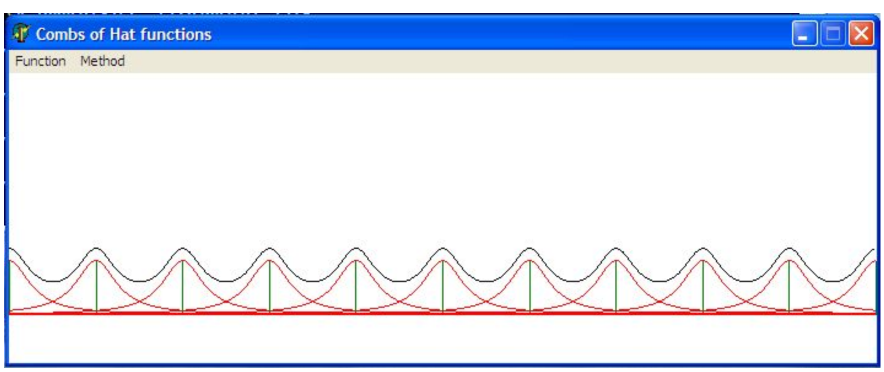
$$= \frac{d/2\pi}{p^2+d^2}\left[\frac{x}{(1+x^2)^{1/2}}\right]_{-\infty}^{+\infty} = \frac{d/2\pi}{p^2+d^2}.2$$

If the equation of the line is given by $ax+by+c = 0$, then the distance p of a point (x, y) to this line is given by a well-known formula as $p = (ax+by+c)/\sqrt(a^2+b^2)$. Herewith the light strength of a fuzzy image of a line is given by:

$$L(x,y) = \frac{d/\pi}{(ax+by+c)^2/(a^2+b^2)+d^2}$$

This function is known from statistics as a *Cauchy distribution*. Again, for $d \to 0$, a crisp picture is obtained and the integral strength of the light is still equal to unity.

Comb of Cauchy Distributions



The subject of our current study is a comb $P(x)$ of normed Cauchy distributions

$p(x)$ at a one dimensional, infinite and equidistant grid with discretization Δ , embedded on the real axis with coordinate x :

$$P(x) = \sum_{L=-\infty}^{+\infty} p(x - L.\Delta)\,\Delta \qquad \text{where} \quad p(x) = \frac{\sigma/\pi}{\sigma^2 + x^2}$$

Check that $p(x)$ is indeed normed:

$$\int_{-\infty}^{+\infty} \frac{\sigma/\pi}{\sigma^2 + x^2}\,dx = \int_{-\infty}^{+\infty} \frac{1/\pi}{1 + (x/\sigma)^2}\,d(x/\sigma) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{dt}{1+t^2} =$$

$$\frac{1}{\pi}\left[\arctan(t)\right]_{-\infty}^{+\infty} = \pi/\pi = 1$$

The spread σ of a Cauchy distribution, despite of its name, is *not* a standard deviation, as is clear from the following, with help of the above:

$$\int_{-\infty}^{+\infty} \frac{\sigma/\pi}{\sigma^2 + x^2}\,x^2\,dx = \frac{\sigma}{\pi}\int_{-\infty}^{+\infty} \frac{\sigma^2 + x^2}{\sigma^2 + x^2}\,dx - \sigma^2 \int_{-\infty}^{+\infty} \frac{\sigma/\pi}{\sigma^2 + x^2}\,dx$$

$$= \frac{\sigma}{\pi}\left[x\right]_{-\infty}^{+\infty} - \sigma^2 \,.\, 1 = \frac{\sigma}{\pi}\,2\,\infty - \sigma^2 = \infty$$

Nothing can remove this infinite outcome for the standard deviation of a Cauchy distribution. The Fourier series for a comb of Cauchy distributions is:

$$P(x) = 1 + 2 \times \sum_{k=1}^{\infty} A(k\omega)\,cos(k\omega x)$$

Where it is reminded that $\omega = 2\pi/\Delta$. And:

$$A(y) = \int_{-\infty}^{+\infty} p(x)\cos(yx)\,dx = \frac{\sigma}{\pi}\int_{-\infty}^{+\infty} \frac{\cos(yx)}{\sigma^2 + x^2}\,dx$$

Complex Analysis is required - oh well: for the moment being - in order to solve for this integral. Here is a web reference for how to do such a thing:

`http://math.fullerton.edu/mathews/c2003/IntegralsTrigImproperMod.html`

However, a full outcome can be found in the (Dutch) book [S.T.M Ackermans & J.H. van Lint, *algebra en analyse*, Academic Service, Den Haag (1976)]. On page 458 we read:

$$\int_0^{\infty} \frac{\cos(x)}{a^2 + x^2}dx = \frac{\pi e^{-a}}{2a}$$

In concordance with this, we shall rewrite the expression for $A(y)$ a little bit:

$$A(y) = \frac{\sigma}{\pi}\int_{-\infty}^{+\infty} \frac{\cos(yx)y}{y^2(\sigma^2 + x^2)}\,d(yx) = \frac{y\sigma}{\pi}\int_{-\infty}^{+\infty} \frac{\cos(x)}{(y\sigma)^2 + x^2}\,dx$$

$$= \frac{a}{\pi} 2 \int_0^{+\infty} \frac{\cos(x)}{a^2 + x^2}\, dx = \frac{a}{\pi}\, 2\, \frac{\pi e^{-a}}{2a} = e^{-a}$$

Where $a = y\sigma$, resulting in an extremely simple outcome for the coefficients: $A(y) = e^{- \ \sigma}$. We conclude that the Fourier series of a Uniform Comb of Cauchy distributions is given by:

$$P(x) = \sum_{L=-\infty}^{+\infty} p(x - L\Delta)\, \Delta = 1 + 2 \times \sum_{k=1}^{\infty} e^{-k\omega\sigma}\, cos(k\omega x)$$

Where it is reminded that: $\omega = 2\pi/\Delta$.

It is seen that $P(x)$ is approximately equal to 1 , provided that the rest of the Fourier series expansion is sufficiently small. First we take, out of thin air, an "acceptable" (relative) *error* $0 < \epsilon < 1$. As a next step, the following requirement is imposed.

$$\left| 2 \times \sum_{k=1}^{\infty} e^{-k\omega\sigma}\, cos(k\omega x) \right| < \epsilon$$

Because the cosine is in between -1 and $+1$, it is furthermore evident that a sufficient condition for the above is:

$$\sum_{k=1}^{\infty} e^{-k\omega\sigma} < \epsilon/2$$

A geometric series is recognized herein:

$$e^{-\omega\sigma} \sum_{k=0}^{\infty} \left(e^{-\omega\sigma}\right)^k < \epsilon/2 \quad \Longleftrightarrow \quad \frac{e^{-\omega\sigma}}{1 - e^{-\omega\sigma}} < \epsilon/2$$

$$\Longleftrightarrow \quad e^{-\omega\sigma} < \frac{\epsilon/2}{1 + \epsilon/2} \quad \Longleftrightarrow \quad \sigma > \frac{\Delta}{2\pi} \ln\left(\frac{1 + \epsilon/2}{\epsilon/2}\right)$$

Therefore the quantity α (alpha) for a Cauchy distribution is defined as:

$$\alpha = \ln\left(1 + \frac{2}{\epsilon}\right) \approx \ln(2/\epsilon) \quad \Longrightarrow \quad \sigma > \frac{\Delta}{2\pi}\alpha$$

The latter approximation because errors are supposed to be small. It is noted that the approximation can also be obtained by considering only the first term of the rest of the Fourier expansion instead of the whole rest of the series. It is concluded that α is varying somewhat less slowly when compared with the analogous quantity for Gauss distributions, which has been found before as $\alpha = \sqrt{2\, \ln(2/\epsilon)}$ in *Uniform Combs of aussians*:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2010/dikte/document.pdf`

Improved Error Analysis

Parceval's Theorem for uniform combs of hat functions $P(x)$ with discretization Δ reads as follows.

$$\frac{1}{\Delta/2}\int_{-\Delta/2}^{+\Delta/2} P(x)^2\,dx = \frac{1}{2}a_0^2 + \sum_{k=1}^{\infty} a_k^2$$

Where $a_k/2 = A(k.\omega)$, $\omega = 2\pi/\Delta$ and:

$$P(x) = \sum_{L=-\infty}^{+\infty} p(x - L\Delta)\Delta = 1 + \sum_{k=1}^{\infty} a_k\, cos(k\omega x)$$

Lemma.

$$\frac{1}{\Delta/2}\int_{-\Delta/2}^{+\Delta/2} P(x)\,dx = 2$$

Proof. In the subsection *Uniform Combs of Hat functions* it is has been shown that:

$$a_k = \frac{1}{\Delta/2}\int_{-\Delta/2}^{+\Delta/2} P(x)\cos(k\,2\pi/\Delta\,x)\,dx = 2\times A(k\omega) \quad \Longrightarrow$$

$$\frac{1}{\Delta/2}\int_{-\Delta/2}^{+\Delta/2} P(x)\,dx = a_0 = 2\times A(0) = 2$$

End of proof. But not the end of error analysis.

$$\frac{1}{2}a_0^2 = 2\,A^2(0) = 2 \quad ; \quad a_k^2 = 4\,A^2(k\omega) \quad \Longrightarrow$$

$$\frac{1}{\Delta/2}\int_{-\Delta/2}^{+\Delta/2} \left[P(x) - 1\right]^2\,dx =$$

$$\frac{1}{\Delta/2}\int_{-\Delta/2}^{+\Delta/2} P(x)^2\,dx - 2\frac{1}{\Delta/2}\int_{-\Delta/2}^{+\Delta/2} P(x)\,dx + \frac{1}{\Delta/2}\int_{-\Delta/2}^{+\Delta/2} 1\,dx =$$

$$2 + 4\sum_{k=1}^{\infty} A^2(k\omega) - 2\times 2 + 2 \quad \Longrightarrow$$

$$\frac{1}{\Delta/2}\int_{-\Delta/2}^{+\Delta/2} \left[P(x) - 1\right]^2\,dx = \sum_{k=1}^{\infty} a_k^2 \qquad \text{where} \quad a_k = 2A(k\omega)$$

In words. The square of the differences between the comb and unity, integrated over a discretization interval (and divided by half of it) is equal to the sum of squares of the Fourier coefficients (of the cosines).
The left hand side can be interpreted as (twice) a mean square relative error, which is, of course, a more accurate measure than just the first term of the remaining Fourier series. The former method is quite good for a *Uniform Comb*

of aussians, but for a uniform comb of e.g. Cauchy distributions, it is already a bit doubtful, as we have seen in the preceding subsection.
Example. Uniform comb of Gaussians. Let:

$$\sum_{k=1}^{\infty} [2 \times A(k\omega)]^2 < \epsilon^2 \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} e^{-(k\omega\sigma)^2} < (\epsilon/2)^2$$

Because this series is converging very fast, we decide again to take only the first term of it:

$$e^{-(\omega\sigma)^2} < (\epsilon/2)^2 \quad \Longleftrightarrow \quad e^{-(\omega\sigma)^2/2} < \epsilon/2$$

Thus resulting in exactly the same condition as found before:

$$\sigma = \frac{\Delta}{2\pi}\alpha \qquad \text{where} \quad \alpha = \sqrt{2\ln(2/\epsilon)}$$

Example. Uniform comb of Cauchy's. Let:

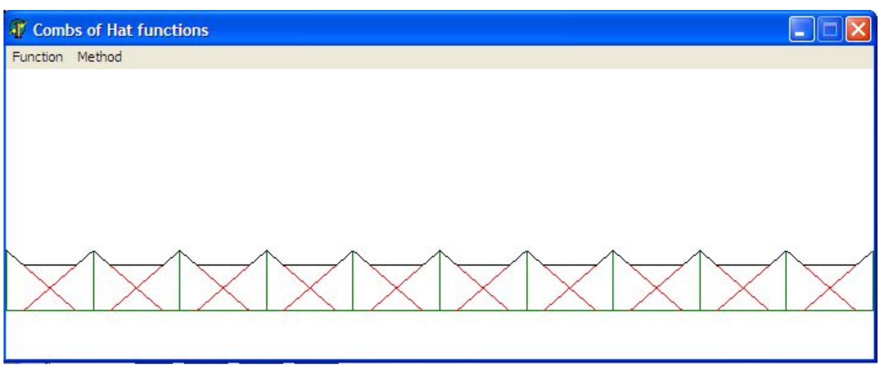
$$\sum_{k=1}^{\infty} [2 \times A(k\omega)]^2 < \epsilon^2 \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} \left[e^{-2\omega\sigma}\right]^k < (\epsilon/2)^2$$

$$\Longleftrightarrow \quad \frac{e^{-2\omega\sigma}}{1 - e^{-2\omega\sigma}} < (\epsilon/2)^2 \quad \Longleftrightarrow \quad e^{-2\omega\sigma} < \frac{(\epsilon/2)^2}{1 + (\epsilon/2)^2}$$

$$\Longleftrightarrow \quad \sigma > \frac{\Delta}{2\pi} \ln\sqrt{(1 + (2/\epsilon)^2} \approx \frac{\Delta}{2\pi}\ln(2/\epsilon)$$

The latter approximation because errors are supposed to be small. Which then is the same as found with the simpler method.

Comb of Triangles



The subject of our current study is a Comb $P(x)$ of triangular distributions:

$$P(x) = \sum_{L=-\infty}^{+\infty} p(x - L.\Delta)\,\Delta$$

$$\text{where}\quad p(x) = \begin{cases} 0 & \text{for} \quad x \le -\sigma \\ (\sigma + x)/\sigma^2 & \text{for} \quad -\sigma \le x \le 0 \\ (\sigma - x)/\sigma^2 & \text{for} \quad 0 \le x \le +\sigma \\ 0 & \text{for} \quad +\sigma \le x \end{cases}$$

The geometrical picture is a triangle with base 2σ and height $1/\sigma$, resulting in an area 1, thus establishing that the function $p(x)$ is normed. The spread $\sigma > 0$ of a triangular distribution, despite of its name, is *not* exactly a standard deviation, as is clear from the following:

$$\int_{-\infty}^{+\infty} x^2 p(x)\, dx = \int_{-\sigma}^{0} x^2 \frac{\sigma + x}{\sigma^2}\, dx + \int_{0}^{+\sigma} x^2 \frac{\sigma - x}{\sigma^2}\, dx =$$

$$\left[\frac{x^3}{3\sigma} + \frac{x^4}{4\sigma^2}\right]_{-\sigma}^{0} + \left[\frac{x^3}{3\sigma} - \frac{x^4}{4\sigma^2}\right]_{0}^{+\sigma} = \sigma^2/3 - \sigma^2/4 + \sigma^2/3 - \sigma^2/4 = \sigma^2/6$$

$$\Longrightarrow \quad \text{standard deviation} = \sigma/\sqrt{6}$$

$P(x)$ is developed into the standard Fourier series for combs of hat functions:

$$P(x) = 1 + 2 \times \sum_{k=1}^{\infty} A(k\omega)\, cos(k\omega x) \qquad \text{where} \quad A(y) = \int_{-\infty}^{+\infty} p(x) \cos(xy)\, dx$$

With a little help from MAPLE - could have done this by hand, but I'm lazy:

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int((sigma+x)/sigma^2*cos(y*x),x=-sigma..0) +
int((sigma-x)/sigma^2*cos(y*x),x=0..+sigma);
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Giving:

$$-\frac{2(-1 + cos(\sigma y))}{\sigma^2 y^2}$$

To be converted into a slightly different expression with $\cos(2x) = 1 - 2\sin^2(x)$:

$$-2\frac{-1 + \cos(y\sigma)}{y^2\sigma^2} = \frac{\sin^2(\frac{1}{2}y\sigma)}{(\frac{1}{2}y\sigma)^2} \qquad \Longrightarrow$$

$$P(x) = 1 + 2 \times \sum_{k=1}^{\infty} \left[\frac{\sin(\frac{1}{2}k\omega\sigma)}{\frac{1}{2}k\omega\sigma}\right]^2 \cos(k\omega x) \qquad \text{where} \quad \omega = \frac{2\pi}{\Delta}$$

The Fourier analysis of a comb of Triangles is somewhat deviant from the Fourier analysis of a comb of Gaussians or Cauchy distributions. For $\sigma = m\Delta$, with $m > 0$ integer, the outcome is simply $P(x) = 1$, without any wiggles or giggles. On the other hand, especially for $\sigma = (m + \frac{1}{2})\Delta$, it's much harder to get rid of those wiggles. Proceeding as usual:

$$\sum_{k=1}^{\infty} \left[\frac{\sin(\frac{1}{2}k\omega\sigma)}{\frac{1}{2}k\omega\sigma}\right]^2 \le \sum_{k=1}^{\infty} \left[\frac{1}{\frac{1}{2}k\omega\sigma}\right]^2 < \epsilon/2 \quad \Longrightarrow \quad \frac{1}{(\frac{1}{2}\omega\sigma)^2} \sum_{k=1}^{\infty} \frac{1}{k^2} < \epsilon/2$$

Calculations can proceed because of a well known sum:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} = \zeta(2) \quad \Longrightarrow \quad \frac{\Delta^2}{\pi^2 \sigma^2} \frac{\pi^2}{6} < \epsilon/2 \quad \Longrightarrow \quad \frac{\Delta^2}{\sigma^2} < 3\,\epsilon$$

$$\Longrightarrow \quad \sigma = \frac{\Delta}{\sqrt{3\,\epsilon}} \qquad \text{or} \qquad \sigma = \frac{\Delta}{2\pi}\,\alpha \quad \text{where} \quad \alpha = \frac{2\pi}{\sqrt{3\,\epsilon}}$$

Let's try the "improved" error analysis:

$$\frac{1}{(\frac{1}{2}\omega\sigma)^4} \sum_{k=1}^{\infty} \frac{1}{k^4} < (\epsilon/2)^2$$

Calculations can proceed because of a well known sum:

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90} = \zeta(4) \quad \Longrightarrow \quad \frac{\Delta^4}{\pi^4 \sigma^4} \frac{\pi^4}{90} < (\epsilon/2)^2 \quad \Longrightarrow \quad \frac{\Delta^4}{\sigma^4} < 90/4\,\epsilon^2$$

Conclusion:

$$\sigma > \frac{\Delta}{\sqrt{\epsilon\sqrt{45/2}}} \approx 0,45915 \frac{\Delta}{\sqrt{\epsilon}} \qquad \text{or} \qquad \sigma > \frac{\Delta}{\sqrt{3\,\epsilon}} \approx 0,57735 \frac{\Delta}{\sqrt{\epsilon}}$$

After all, it's only a matter of error *estimates*, no more, no less.

Comb of Rectangles

The subject of our current study is a Comb $P(x)$ of rectangular distributions:

$$P(x) = \sum_{L=-\infty}^{+\infty} p(x - L.\Delta)\,\Delta$$

$$\text{where} \quad p(x) = \begin{cases} 0 & \text{for} \quad x \le -\frac{1}{2}\sigma \\ 1/\sigma & \text{for} \quad -\frac{1}{2}\sigma \le x \le +\frac{1}{2}\sigma \\ 0 & \text{for} \quad +\frac{1}{2}\sigma \le x \end{cases}$$

The geometry of this is a rectangle with width σ and height $1/\sigma$, resulting in an area 1, thus establishing that the function $p(x)$ is normed. The spread $\sigma > 0$ of a rectangular distribution, despite of its name, is *not* exactly a standard deviation, as is clear from the following:

$$\int_{-\infty}^{+\infty} x^2 p(x)\,dx = \int_{-\frac{1}{2}\sigma}^{+\frac{1}{2}\sigma} x^2/\sigma\,dx = \left[\frac{x^3}{3\sigma}\right]_{-\frac{1}{2}\sigma}^{+\frac{1}{2}\sigma} = \frac{\sigma^2}{8\times 3}\times 2$$

$$\Longrightarrow \quad \text{standard deviation} = \sigma/(2\sqrt{3})$$

$P(x)$ is developed into the standard Fourier series for combs of hat functions:

$$P(x) = 1 + 2\times\sum_{k=1}^{\infty} A(k\omega)\,cos(k\omega x) \qquad \text{where} \quad A(y) = \int_{-\infty}^{+\infty} p(x)\cos(xy)\,dx$$

By hand, because I'm not too lazy:

$$A(y) = \int_{-\frac{1}{2}\sigma}^{+\frac{1}{2}\sigma} 1/\sigma\cos(xy)\,dx = \left[\frac{\sin(yx)}{y\sigma}\right]_{-\frac{1}{2}\sigma}^{+\frac{1}{2}\sigma} = \frac{\sin(\frac{1}{2}y\sigma)}{\frac{1}{2}y\sigma} = \text{sinc}(\frac{1}{2}y\sigma) \quad \Longrightarrow$$

$$P(x) = 1 + 2\times\sum_{k=1}^{\infty}\frac{\sin(\frac{1}{2}k\omega\sigma)}{\frac{1}{2}k\omega\sigma}\cos(k\omega x) \qquad \text{where} \quad \omega = \frac{2\pi}{\Delta}$$

The Fourier analysis of a comb of Rectangles is thus similar to the Fourier analysis of a comb of Triangles. For $\sigma = m\Delta$, with $m > 0$ integer, the outcome is simply $P(x) = 1$, without any wiggles. On the other hand, especially for $\sigma = (m + \frac{1}{2})\Delta$, it's much harder to get rid of those wiggles. Furthermore, proceeding as usual is virtually impossible because of the terms with $1/k$, together with the fact that the harmonic series is known to be divergent:

$$\sum_{k=1}^{\infty}\frac{1}{k} = \infty$$

So it seems that the only possibility left is our *Improved Error Analysis*:

$$\sum_{k=1}^{\infty}\left[\frac{\sin(\frac{1}{2}k\omega\sigma)}{\frac{1}{2}k\omega\sigma}\right]^2 < (\epsilon/2)^2$$

It is seen that the left hand side of this expression is exactly the same as the left hand side of the normal error analysis for triangular distributions. Therefore we can immediately conclude that:

$$\sigma > \frac{\Delta}{\sqrt{3/2}\,\epsilon}$$

Find more Fourier integrals

High on the wish list of people who find Laplace transforms interesting is the Laplace transform of a sine function:

$$\int_0^\infty e^{-sx}\sin(x)\,dx = ??$$

We proceed by integrating by parts:

$$\int_0^\infty e^{-sx}\sin(x)\,dx = \int_0^\infty e^{-sx} d\,[-\cos(x)] =$$

$$-\left[e^{-sx}\cos(x)\right]_0^\infty + \int_0^\infty \cos(x)\,d\left[e^{-sx}\right] = 1 - s\int_0^\infty e^{-sx}\cos(x)\,dx$$

Feels good. Let's do it again:

$$1 - s\int_0^\infty e^{-sx}\cos(x)\,dx = 1 - s\int_0^\infty e^{-sx} d\,[\sin(x)] =$$

$$1 - s\left[e^{-sx}\sin(x)\right]_0^\infty + s\int_0^\infty \sin(x)\,d\left[e^{-sx}\right] =$$

$$1 - s^2\int_0^\infty e^{-sx}\sin(x)\,dx = \int_0^\infty e^{-sx}\sin(x)\,dx \quad \Longrightarrow$$

Theorem. Laplace transform of sine:

$$\int_0^\infty e^{-sx}\sin(x)\,dx = \frac{1}{1+s^2}$$

Theorem. Laplace transform of cosine:

$$\int_0^\infty e^{-sx}\cos(x)\,dx = \frac{s}{1+s^2}$$

Proof.

$$\int_0^\infty e^{-sx}\cos(x)\,dx = \int_0^\infty e^{-sx}\,d\sin(x) = \left[e^{-sx}\sin(x)\right]_0^\infty - \int_0^\infty \sin(x)\,de^{-sx}$$

$$= 0 + s\int_0^\infty e^{-sx}\sin(x)\,dx = \frac{s}{1+s^2}$$

Theorem. Fourier integral of (normed) decay function:

$$\int_{-\infty}^{+\infty} \frac{1}{2}e^{-|x|}\cos(yx)\,dx = \frac{1}{1+y^2}$$

Proof.

$$\int_0^\infty e^{-sx}\cos(x)\,dx = \frac{s}{1+s^2} \quad \Longrightarrow \quad \int_{-\infty}^{+\infty} \frac{1}{2}e^{-|x|}\cos(yx)\,dx =$$

$$1/y \int_0^{\infty} e^{-(\ \ x)/\ \ } \cos(yx)\, d(yx) = 1/y \, \frac{1/y}{1+(1/y)^2} = \frac{1}{1+y^2}$$

Theorem. Half area of sinc function:

$$\int_0^{\infty} \operatorname{sinc}(x)\, dx = \frac{\pi}{2} \quad \Longrightarrow \quad \text{normed } = \frac{1}{\pi} \operatorname{sinc}(x)$$

Proof.

$$\int_0^{\infty} \left[\int_0^{\infty} e^{-sx} \sin(x)\, dx \right] ds = \int_0^{\infty} \left[\int_0^{\infty} e^{-sx} \sin(x)\, ds \right] dx$$

Because iterated limits in the real world do always commute. Now we have a left hand side and a right hand side. The left hand side, according to the above, is equal to:

$$\int_0^{\infty} \left[\int_0^{\infty} e^{-sx} \sin(x)\, dx \right] ds = \int_0^{\infty} \frac{ds}{1+s^2} = [\arctan(s)]_0^{\infty} = \frac{\pi}{2}$$

The right hand side, on the other hand:

$$\int_0^{\infty} \left[\int_0^{\infty} e^{-sx} \sin(x)\, ds \right] dx = \int_0^{\infty} \sin(x) \left[\int_0^{\infty} e^{-sx}\, ds \right] dx =$$

$$\int_0^{\infty} \sin(x) \frac{1}{x} \left[-e^{-\ \ } \right]_0^{\infty} dx = \int_0^{\infty} \frac{\sin(x)}{x}\, dx \qquad \text{Q.E.D.}$$

Theorem. Fourier integral of sinc function:

$$\int_{-\infty}^{+\infty} \operatorname{sinc}(x) \cos(yx)\, dx = A(y)$$

Where:

$$A(y) = \begin{cases} 0 & \text{for} \quad y < -1 \\ \pi & \text{for} \quad -1 < y < +1 \\ 0 & \text{for} \quad +1 < y \end{cases}$$

Proof.

$$2 \sin(\alpha) \cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \quad \Longrightarrow$$

$$\int_{-\infty}^{+\infty} \frac{\sin(x)\cos(yx)}{x}\, dx = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\sin(x+yx)}{x}\, dx + \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\sin(x-yx)}{x}\, dx$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\sin((1+y)x)}{(1+y)x} (1+y)\, dx + \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\sin((1-y)x)}{(1-y)x} (1-y)\, dx$$

Now let $u = (1+y)x$ and $v = (1-y)x$. Then:

$$\int_{-\infty}^{+\infty} \frac{\sin((1+y)x)}{(1+y)x} (1+y)\, dx = \begin{cases} + \int_{-\infty}^{+\infty} \operatorname{sinc}(u)\, du & \text{for} \quad 1+y > 0 \\ - \int_{-\infty}^{+\infty} \operatorname{sinc}(u)\, du & \text{for} \quad 1+y < 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} \frac{\sin((1-y)x)}{(1-y)x}(1-y)\,dx = \begin{cases} +\int_{-\infty}^{+\infty} \operatorname{sinc}(v)\,dv & \text{for} \quad 1-y>0 \\ -\int_{-\infty}^{+\infty} \operatorname{sinc}(v)\,dv & \text{for} \quad 1-y<0 \end{cases}$$

Where $\int_{-\infty}^{+\infty} \operatorname{sinc}(u)\,du = 2\int_0^{\infty} \operatorname{sinc}(v)\,dv = 2.\frac{1}{2}\pi = \pi$. Summarizing:

	$\int_{-\infty}^{+\infty} \operatorname{sinc}(u)\,du$	$\int_{-\infty}^{+\infty} \operatorname{sinc}(v)\,dv$	$\int_{-\infty}^{+\infty} \operatorname{sinc}(x)\cos(yx)\,dx$
$y < -1$	$-\pi$	$+\pi$	0
$-1 < y < +1$	$+\pi$	$+\pi$	π
$+1 < y$	$+\pi$	$-\pi$	0

Theorem. Parceval's theorem for sinc function:

$$\int_{-\infty}^{+\infty} \operatorname{sinc}^2(x)\,dx = \pi \quad \Longrightarrow \quad \text{normed} = \frac{1}{\pi}\operatorname{sinc}^2(x)$$

Proof. The integral of the square of the Fourier transform of a function is equal yox the)integral of the square of the function itself.9738Tf9.962-2.4Theorem.

Where we have put $\pi x/\sigma = \xi$. The latter outcome is equal to a block function $_\sigma(y)$, where:

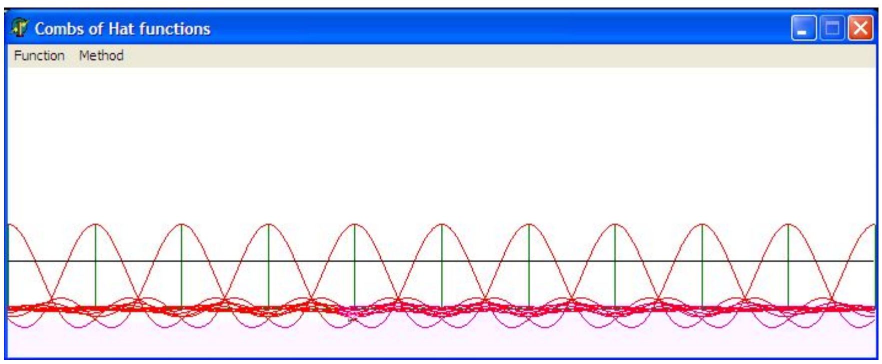
$$ {}_\sigma(y) = \left\{ \begin{array}{lll} 0 & \mbox{for} & \sigma/\pi y < -1 \\ \pi/\pi & \mbox{for} & -1 < \sigma/\pi y < +1 \\ 0 & \mbox{for} & +1 < \sigma/\pi y \end{array} \right. $$

Written otherwise:

$$ {}_\sigma(y) = \left\{ \begin{array}{lll} 0 & \mbox{for} & y < -\pi/\sigma \\ 1 & \mbox{for} & -\pi/\sigma < y < +\pi/\sigma \\ 0 & \mbox{for} & +\pi/\sigma < y \end{array} \right. $$

Conclusion - remember that $\omega = 2\pi/\Delta$:

$$ \sum_{L=-\infty}^{+\infty} \frac{\Delta}{\sigma} \operatorname{sinc}\left(\frac{\pi}{\sigma}\left[x - L\Delta\right]\right) = 1 + 2 \times \sum_{k=1}^{\infty} {}_\sigma(k\omega)\cos(k\omega x) $$



The right hand side is *exactly* equal to one iff for all $k > 1$:

$$ +\pi/\sigma < k\omega \quad \Longleftrightarrow \quad \pi\frac{\Delta}{2\pi} < \sigma \quad \Longleftrightarrow \quad \sigma > \frac{\Delta}{2} $$

A true miracle has happened. There is *no error* present, at all, in the following formula. Which thus holds *exactly, for any* $\sigma > \Delta/2$:

$$ \sum_{k=-\infty}^{+\infty} \frac{\Delta}{\sigma} \operatorname{sinc}\left(\frac{\pi}{\sigma}\left[x - k\Delta\right]\right) = 1 $$

Where have we seen this before .. ? Let $f(x) = 1$ and $f_k = f(k\Delta)$ and do a trivial rewrite:

$$ \sum_{k=-\infty}^{+\infty} \frac{\Delta}{\sigma} f_k \operatorname{sinc}\left(\frac{\pi}{\sigma}\left[x - k\Delta\right]\right) = f(x) $$

Now take a look at *Shannon's Sampling Theorem*, for example at:

`http://en.wikipedia.org/wiki/Nyquist%E2%80%93Shannon_sampling_theorem`

It should be obvious that there exists a correspondence with our own theory. It is also clear that the *Special Theory of Continuity* can, in principle, be modified as to include general function behaviour as well: $f(x) \neq 1$. And there is a section *Nonuniform sampling* in that web page, where we can read about a second item on our To Do list: the sampling theory of Shannon can be generalized for the case of nonuniform samples.

Inverse Fourier transform method

We have found that the Fourier transform of a block function is a sinc and the Fourier transform of a sinc is a block function. This is, of course, no coincidence. Hat functions are symmetric, therefore the Fourier integral and the inverse thereof are:

$$\int_{-\infty}^{+\infty} p(x)\cos(yx)\,dx = A(y) \quad\Longleftrightarrow\quad \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(y)\cos(xy)\,dy = p(x)$$

So if the Fourier integral of a hat function $p(x)$ is $A(y)$, then the Fourier integral of the hat function $A(x)$ is $2\pi p(y)$. Let's check it at hand of functions in previous subsections:

$$p(x) = \frac{1}{\sigma}\,\text{sinc}\left(\frac{\pi}{\sigma}x\right) \quad\Longleftrightarrow\quad A(y) = \begin{cases} 0 & \text{for} \quad y < -\pi/\sigma \\ 1 & \text{for} \quad -\pi/\sigma < y < +\pi/\sigma \\ 0 & \text{for} \quad +\pi/\sigma < y \end{cases}$$

$$p(x) = \begin{cases} 0 & \text{for} \quad x \le -\frac{1}{2}\sigma \\ 1/\sigma & \text{for} \quad -\frac{1}{2}\sigma \le x \le +\frac{1}{2}\sigma \\ 0 & \text{for} \quad +\frac{1}{2}\sigma \le x \end{cases} \quad\Longleftrightarrow\quad A(y) = \text{sinc}(\frac{1}{2}y\sigma)$$

Modify the latter formulas by the method of careful substitution. Three steps can be distinguished in this procedure: (1) adjust proper spread, (2) adjust proper norm, (3) what's in a name? Okay, let's just do it. Start with adjusting proper spread and let $\sigma/2 \to \pi/\sigma$:

$$p(x) = \begin{cases} 0 & \text{for} \quad x \le -\pi/\sigma \\ \sigma/(2\pi) & \text{for} \quad -\pi/\sigma \le x \le +\pi/\sigma \\ 0 & \text{for} \quad +\pi/\sigma \le x \end{cases} \quad\Longleftrightarrow\quad A(y) = \text{sinc}(y\pi/\sigma)$$

Now adjust proper norm by multipling both right hand sides with $2\pi/\sigma$:

$$p(x) = \begin{cases} 0 & \text{for} \quad x \le -\pi/\sigma \\ 1 & \text{for} \quad -\pi/\sigma \le x \le +\pi/\sigma \\ 0 & \text{for} \quad +\pi/\sigma \le x \end{cases} \quad\Longleftrightarrow\quad A(y) = 2\pi \times \frac{1}{\sigma}\,\text{sinc}(y\pi/\sigma)$$

At last, what's in a name? By $x \leftrightarrow y$ and $p \leftrightarrow A$, where, according to the inverse Fourier transform method, the factor (2π) must be discarded:

$$p(x) = \frac{1}{\sigma}\,\text{sinc}\left(\frac{\pi}{\sigma}x\right) \quad\Longleftrightarrow\quad A(y) = \begin{cases} 0 & \text{for} \quad y < -\pi/\sigma \\ 1 & \text{for} \quad -\pi/\sigma < y < +\pi/\sigma \\ 0 & \text{for} \quad +\pi/\sigma < y \end{cases}$$

Quod Erat Demonstrandum / Quite Easily Done.
Slightly more interesting is to apply the method for obtaining a new result, instead of reproducing an old one. Copy and paste from the subsection *Comb of Triangles*:

$$p(x) = \begin{cases} 0 & \text{for} \quad x \le -\sigma \\ (\sigma + x)/\sigma^2 & \text{for} \quad -\sigma \le x \le 0 \\ (\sigma - x)/\sigma^2 & \text{for} \quad 0 \le x \le +\sigma \\ 0 & \text{for} \quad +\sigma \le x \end{cases} \quad \Longleftrightarrow \quad A(y) = \operatorname{sinc}^2(\frac{1}{2} y\sigma)$$

Modify the latter formulas by the method of careful substitution. Adjust proper spread first and let $\sigma/2 \to \pi/\sigma$ again:

$$p(x) = \frac{\sigma}{2\pi} \begin{cases} 0 & \text{for} \quad x \le -2\pi/\sigma \\ 1 + x\sigma/(2\pi) & \text{for} \quad -2\pi/\sigma \le x \le 0 \\ 1 - x\sigma/(2\pi) & \text{for} \quad 0 \le x \le +2\pi/\sigma \\ 0 & \text{for} \quad +2\pi/\sigma \le x \end{cases} \quad \Longleftrightarrow \quad A(y) = \operatorname{sinc}^2(y\pi/\sigma)$$

Now adjust proper norm by multipling both right hand sides with $2\pi/\sigma$:

$$p(x) = \begin{cases} 0 & \text{for} \quad x \le -2\pi/\sigma \\ 1 + x\sigma/(2\pi) & \text{for} \quad -2\pi/\sigma \le x \le 0 \\ 1 - x\sigma/(2\pi) & \text{for} \quad 0 \le x \le +2\pi/\sigma \\ 0 & \text{for} \quad +2\pi/\sigma \le x \end{cases} \quad \Longleftrightarrow \quad A(y) = \frac{2\pi}{\sigma} \operatorname{sinc}^2(y\pi/\sigma)$$

Now in the space domain $p(0) = 1$, which is the same as, in the Fourier domain:

$$\frac{2\pi}{\sigma} \int_{-\infty}^{+\infty} \operatorname{sinc}^2(y\pi/\sigma)\, dy = 2 \int_{-\infty}^{+\infty} \operatorname{sinc}^2(y\pi/\sigma)\, d(y\pi/\sigma) = 2\,\pi$$

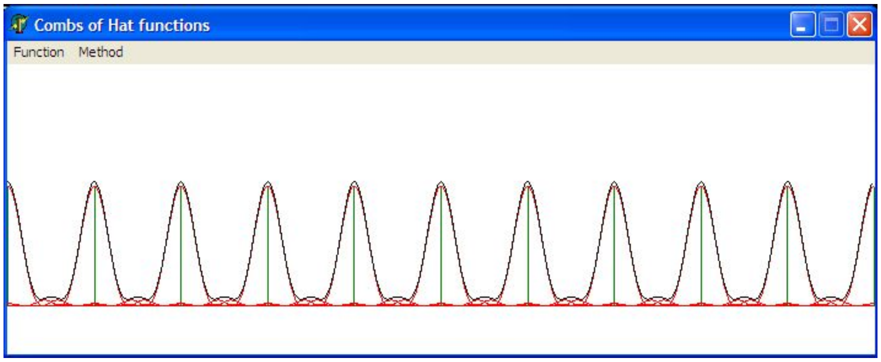
At last, what's in a name? By $x \leftrightarrow y$ and $p \leftrightarrow A$, where, according to the inverse Fourier transform method, the factor (2π) must be discarded:

$$p(x) = \frac{1}{\sigma} \left[\frac{\sin(\pi/\sigma \,.\, x)}{\pi/\sigma \,.\, x} \right]^2 = \frac{1}{\sigma} \operatorname{sinc}^2 \left(\frac{\pi}{\sigma} x \right)$$

$$\Longleftrightarrow \quad A(y) = \begin{cases} 0 & \text{for} \quad y \le -2\pi/\sigma \\ 1 + y\sigma/(2\pi) & \text{for} \quad -2\pi/\sigma \le y \le 0 \\ 1 - y\sigma/(2\pi) & \text{for} \quad 0 \le y \le +2\pi/\sigma \\ 0 & \text{for} \quad +2\pi/\sigma \le y \end{cases}$$

Resulting in a *Comb of Squared Sinc functions* that we haven't seen before:

$$\frac{\Delta}{\sigma} \sum_{L=-\infty}^{+\infty} \operatorname{sinc}^2 \left(\frac{\pi}{\sigma} [x - L\Delta] \right) = 1 + \sum_{k=1}^{\infty} A(k\omega\sigma)\, \cos(k\omega x)$$



So what we have now is the rectangle and the sinc function, the triangle and the sinc squared function. We also have a Cauchy distribution and exponential decay:

$$p(x) = \frac{\sigma/\pi}{\sigma^2 + x^2} \quad \Longleftrightarrow \quad A(y) = e^{-|\;|\sigma}$$

The absolute value is needed because we must have a *hat* shape and at the same time prevent an explosion for negative y . The other way around; adjust proper spread first. For an exponential decay that is $\exp(-|y|/\sigma)$, where σ is the decay rate. Therefore substitute $\sigma \to 1/\sigma$, resulting in:

$$p(x) = \frac{\sigma/\pi}{1 + (x\sigma)^2} \quad \Longleftrightarrow \quad A(y) = e^{-|\;|/\sigma}$$

Now adjust proper norm by multipling both right hand sides with π/σ :

$$p(x) = \frac{1}{1 + (x\sigma)^2} \quad \Longleftrightarrow \quad A(y) = 2\pi \times \frac{e^{-|\;|/\sigma}}{2\sigma}$$

At last, what's in a name? By $x \leftrightarrow y$ and $p \leftrightarrow A$, where, according to the inverse Fourier transform method, the factor (2π) must be discarded:

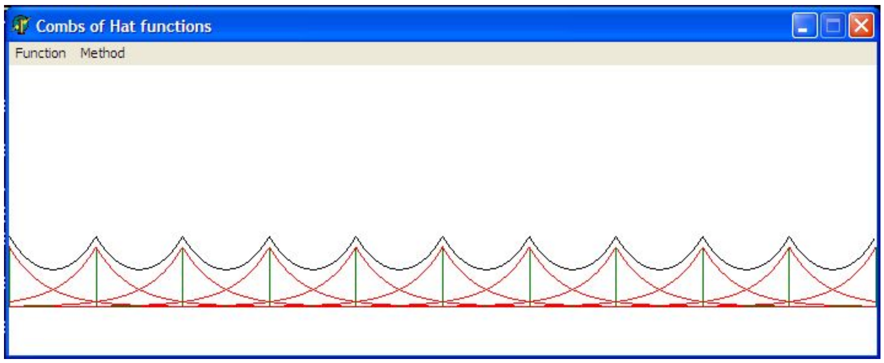
$$p(x) = \frac{e^{-|x|/\sigma}}{2\sigma} \quad \Longleftrightarrow \quad A(y) = \frac{1}{1 + (y\sigma)^2}$$

Resulting in a *Comb of Exponential Decays* that we haven't seen before:

$$\frac{\Delta}{2\sigma} \sum_{L=-\infty}^{+\infty} e^{-|x-L\Delta|/\sigma} = 1 + \sum_{k=1}^{\infty} \frac{\cos(k\omega x)}{1 + (k\omega\sigma)^2}$$

There are a hundred ways to Rome, though. We could have found this result in a far more direct way with integrals from the subsection *Find more Fourier integrals*.
Working the other way around - that is: *starting* with exponential decays and with the inverse Fourier transform method find the Cauchy distribution - we

could have avoided the complex analysis solution employed in the subsection *Comb of Cauchy Distributions*.



Last but not least, here is our beloved Gaussian:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(x/\sigma)^2} \quad \Longleftrightarrow \quad A(y) = e^{-\frac{1}{2}(\ \sigma)^2}$$

Proper spread with $\sigma \to 1/\sigma$:

$$p(x) = \frac{\sigma}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(x\sigma)^2} \quad \Longleftrightarrow \quad A(y) = e^{-\frac{1}{2}(\ /\sigma)^2}$$

Proper norm with times $\sqrt(2\pi)/\sigma$:

$$p(x) = e^{-\frac{1}{2}(x\sigma)^2} \quad \Longleftrightarrow \quad A(y) = \frac{\sqrt{2\pi}}{\sigma}\, e^{-\frac{1}{2}(\ /\sigma)^2} = 2\pi \times \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(\ /\sigma)^2}$$

At last, what's in a name? By $x \leftrightarrow y$ and $p \leftrightarrow A$, where, according to the inverse Fourier transform method, the factor (2π) must be discarded:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(x/\sigma)^2} \quad \Longleftrightarrow \quad A(y) = e^{-\frac{1}{2}(\ \sigma)^2}$$

Herewith, everything considered so far has been covered: Gaussian $\leftrightarrow$ itself, Cauchy $\leftrightarrow$ Decay, Block $\leftrightarrow$ Shannon, Triangle $\leftrightarrow$ Squared.

Special Theory Finishing Touches

So far so good about the General perspective, but the *Special* Theory is still in the need of some finishing touches. One of the first is that the above interpolation with sinc functions is by far not the only possibility to have a (theoretically) error free recovery of a function from its samples. In fact, *any* hat function with a band limited Fourier transform will do the job.
Such another bandlimited hat function has been found in the subsection *Inverse Fourier transform method*. It is:

$$p(x) = \frac{1}{\sigma}\left[\frac{\sin(\pi/\sigma\,.\,x)}{\pi/\sigma\,.\,x}\right]^2 = \frac{1}{\sigma}\,\text{sinc}^2\left(\frac{\pi}{\sigma}x\right)$$

$$\Longleftrightarrow \quad A(y) = \begin{cases} 0 & \text{for} \quad y \le -2\pi/\sigma \\ 1 + y\sigma/(2\pi) & \text{for} \quad -2\pi/\sigma \le y \le 0 \\ 1 - y\sigma/(2\pi) & \text{for} \quad 0 \le y \le +2\pi/\sigma \\ 0 & \text{for} \quad +2\pi/\sigma \le y \end{cases}$$

Resulting in this comb of squared sinc functions:

$$\frac{\Delta}{\sigma} \sum_{L=-\infty}^{+\infty} \text{sinc}^2 \left(\frac{\pi}{\sigma} [x - L\Delta] \right) = 1 + \sum_{k=1}^{\infty} A(k\omega) \cos(k\omega x)$$

Because $A(y)$ is a triangle which is zero for $|y| \le 2\pi/\sigma$, it is clear that $A(k\omega) = 0$ for all $k > 0$ iff:

$$A(\omega) = 0 \quad \Longleftrightarrow \quad \frac{2\pi}{\Delta} \le \frac{2\pi}{\sigma} \quad \Longleftrightarrow \quad \sigma \ge \Delta$$

And the true miracle has happened again. There is *no error* present, at all, in the following formula. Which thus holds *exactly, for any* $\sigma \ge \Delta$:

$$\sum_{k=-\infty}^{+\infty} \frac{\Delta}{\sigma} \text{sinc}^2 \left(\frac{\pi}{\sigma} [x - k\Delta] \right) = 1$$

And we are not finished yet. As I have said, *any* hat function with a band limited Fourier transform will do the job. There is a thread in the Usenet / Google newsgroup *sci.math* written by this author and called *Sum of inverse cubes*. Yes, that's what you can do with combs of hat functions ! The reference - and my fooling around with the illusion of a great new discovery - can be found in the mathematics newsgroup at:

`http://groups.google.nl/group/sci.math/msg/112107155baa4df7`

Note. The result would only have been brand new if a closed expression would have been found for e.g. $\sum_{k=1}^{\infty} 1/k^3$. This is "a bit" different from the discovery at hand, which on the contrary is a well known result.

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch.