Uniform Combs of Gaussians

Author: Han de Bruijn
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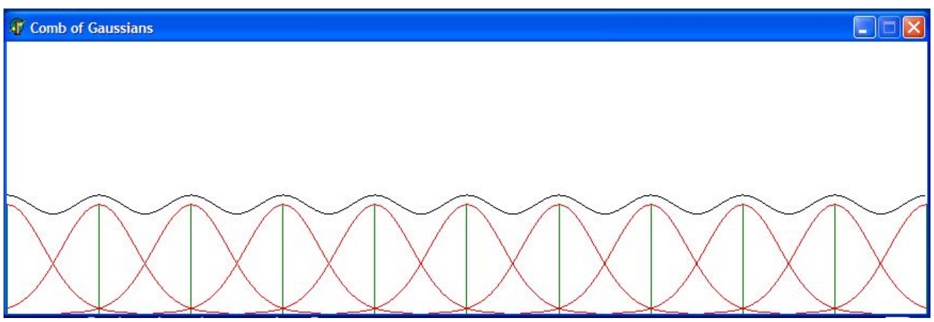
The subject of our current study are Combs of Bell shaped curves $P(x)$. A special but relevant case will be considered in the first place, namely Gauss curves with a spread σ , on a one dimensional, infinite and equidistant grid with discretization Δ , embedded on the real axis with coordinate x :

$$P(x) = \sum_{L=-\infty}^{+\infty} e^{-\frac{1}{2}[(x-L.\Delta)/\sigma]^2}$$

The function $P(x)$ can be interpreted as an attempt to "smooth" the uniform density $x_L = L.\Delta$. Or "make fuzzy" the discretization $f(x_L) = 1$ of a constant - and continuous - function $f(x) = 1$.



As calculated by (version 0.1 of):

`http://hdebruijn.soo.dto.tudelft.nl/jaar2010/dikte/Project5.exe`

Fourier Series

It is easily shown that the above function is *periodic*. Its period is equal to Δ: $P(x+\Delta) = P(x)$ for arbitrary x. This means that $P(x)$ can be developed into a *Fourier series*. In addition, the function is *even*, meaning that $P(x) = P(-x)$, which results in real-valued Fourier coefficients A_k. They are calculated initially as complex-valued entities.

$$A_k + iB_k = \frac{1}{\Delta/2} \int_{-\Delta/2}^{+\Delta/2} P(x) e^{i\,k\,2\pi/\Delta\,.\,x} dx =$$

In the sequel, kind of an angular frequency ω will stand for the quantity $= 2\pi/\Delta$. Then let the calculations continue:

$$= \frac{1}{\Delta/2} \int_{-\Delta/2}^{+\Delta/2} \sum_{L=-\infty}^{+\infty} e^{-(x-L\Delta)^2/2\sigma^2} e^{ik\omega x} dx =$$

$$\frac{1}{\Delta/2}\sum_{L=-\infty}^{+\infty}\int_{-\Delta/2}^{+\Delta/2} e^{-(x-L\Delta)^2/2\sigma^2}e^{ik\omega x}dx$$

Substitute $y = x - L\Delta$ and integrate to y:

$$A_k + iB_k = \frac{1}{\Delta/2}\sum_{L=-\infty}^{+\infty}\int_{-\Delta/2-L\Delta}^{+\Delta/2-L\Delta} e^{-y^2/2\sigma^2}e^{ik\omega(y+L\Delta)}dy =$$

Where:

$$e^{ik\omega(y+L\Delta)} = e^{ik\omega y}e^{ikL2\pi} = e^{ik\omega y}.1$$

Next replace y by $-y$ and switch integration bounds:

$$A_k + iB_k = \frac{1}{\Delta/2}\sum_{L=-\infty}^{+\infty}\int_{L\Delta-\Delta/2}^{L\Delta+\Delta/2} e^{-y^2/2\sigma^2}e^{ik\omega(-y)}dy$$

The above integrals are precisely the adjacent pieces of another integral which has bounds reaching to infinity. That is, they sum up to an infinite integral:

$$A_k + iB_k = \frac{1}{\Delta/2}\int_{-\infty}^{+\infty} e^{-y^2/2\sigma^2}e^{-ik\omega y}dy =$$

$$\frac{1}{\Delta/2}\int_{-\infty}^{+\infty} e^{-y^2/2\sigma^2-ik\omega y}$$

The argument of the exponential function can be written as follows:

$$-y^2/2\sigma^2 - ik\omega y = -\frac{1}{2}(y^2/\sigma^2 - 2.ik\omega\sigma.y/\sigma) =$$

$$-\frac{1}{2}\{y^2/\sigma^2 - 2.ik\omega\sigma.y/\sigma + (ik\omega\sigma)^2\} + \frac{1}{2}(ik\omega\sigma)^2 =$$

$$-(y/\sigma - ik\omega\sigma)^2/2 - (k\omega\sigma)^2/2$$

Resulting in:

$$\frac{1}{\Delta/2}\int_{-\infty}^{+\infty} e^{-(y/\sigma-ik\omega\sigma)^2/2}e^{-(k\omega\sigma)^2/2}\,dy =$$

$$e^{-(k\omega\sigma)^2/2}\frac{1}{\Delta/2}\int_{-\infty}^{+\infty} e^{-(y-ik\omega\sigma^2)^2/2\sigma^2}\,dy$$

We know that the integral is equal to $\sigma\sqrt{2\pi}$, giving at last:

$$A_k + iB_k = A_k = \frac{\sigma\sqrt{2\pi}}{\Delta/2}e^{-(k\omega\sigma)^2/2}$$

The Fourier series of any periodic function is given by:

$$P(x) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k\,cos(k\omega x)$$

We conclude that the Fourier series of a Uniform Comb of Gaussians is given by:

$$P(x) = \sum_{L=-\infty}^{+\infty} e^{-(x-L\Delta)^2/2\sigma^2} = \sigma\sqrt{2\pi}\left[\frac{1}{\Delta} + \frac{1}{\Delta/2}\sum_{k=1}^{\infty} e^{-(k\omega\sigma)^2/2}\, cos(k\omega x)\right]$$

Where it is reminded that: $\omega = 2\pi/\Delta$.

Continuization

As has been said, the Uniform Comb of Gaussians can - or rather maybe should - be interpreted as an attempt to "make continuous again" the discrete function $p(x_L = L.\Delta) = 1$. The latter may be considered as the discretization or *sampling* of a continuous (constant) function: $p(x) = 1$. Thinking along these lines, a natural question is "how good" the *continuization* achieved by the Comb of Gaussians will be, depending on the sampling frequency $\omega = 2\pi/\Delta$ and the spread σ . It is expected that the Fourier Analysis of the preceding section will give us kind of a clue about it:

$$P(x) = \sum_{L=-\infty}^{+\infty} e^{-(x-L\Delta)^2/2\sigma^2} = \sigma\sqrt{2\pi}\left[\frac{1}{\Delta} + \frac{1}{\Delta/2}\sum_{k=1}^{\infty} e^{-(k\omega\sigma)^2/2}\, cos(k\omega x)\right]$$

It is seen that $P(x)$ is approximately equal to a constant, indeed, namely $\sigma\sqrt{2\pi}.1/\Delta$, provided that the next term is sufficiently small. First we take, out of thin air, an "acceptable" (relative) *error*, called $\epsilon(> 0)$. As the next step, we require that the (amplitude of) the second term divided by the first is smaller than the error:

$$\frac{\sigma\sqrt{2\pi}\frac{1}{\Delta/2}e^{-(1.\omega\sigma)^2/2}}{\sigma\sqrt{2\pi}.1/\Delta} = 2\,e^{-(\omega\sigma)^2/2} < \epsilon \quad\Longrightarrow\quad -\frac{1}{2}(\omega\sigma)^2 < \ln(\epsilon/2)$$

$$\Longrightarrow\quad (\frac{2\pi}{\Delta}\sigma)^2 > 2\ \ln(2/\epsilon) \quad\Longrightarrow\quad \sigma > \frac{\Delta}{2\pi}\sqrt{2\ \ln(2/\epsilon)}$$

We define the quantity α (alpha) as:

$$\alpha = \sqrt{2\ \ln(2/\epsilon)} \quad\Longrightarrow\quad \sigma > \frac{\Delta}{2\pi}\alpha$$

Happily for everybody, α is varying *very slowly* as a function of ϵ . Therefore it's *not* necessary to determine that error very precise, in order to have a quite good estimate of the spread σ in the Gaussians. The typical thing is, though, that we really *need* a finite error. For if ϵ could be zero, then α would become infinite and the whole theory would evaporate into nothingness. It's impossible to make the discrete continuous in a mathematics without errors.
Out of thin air .. When visualizing the results of such a "fuzzyfication" as a gray valued image, the Comb is always normed, meaning that all values are

divided by their maximum value. Thus maximum values are made equal to $= 1$, corresponding with black pixels. A natural error with such gray valued images is the lowest increment in grayness, which on a scale ranging from 0 to 1 is equal to $1/256$. Hence, in this case: $\alpha = \sqrt{2\,\ln(512)} \approx 3.53223$, a value which is used throughout in the software accompanying this article.
Another issue is computational efficiency with summing the terms of the Comb. So let's investigate where the values of a Gaussian can be neglected, when compared with our acceptable error (divided by 2 for even better accuracy - we employ the fact that α is rather insensitive to precise values and we prefer to define it once and forever):

$$e^{-\frac{1}{2}[(x-L.\Delta)/\sigma]^2} < \epsilon/2 \quad \Longrightarrow \quad [(x-L.\Delta)/\sigma]^2 > 2\,\ln(2/\epsilon) \quad \Longrightarrow$$

$$|x - L.\Delta| > \sigma\sqrt{2\,\ln(2/\epsilon)} = \sigma\alpha$$

This means that only a neighborhood $|x - L.\Delta| < \sigma\alpha$ around $x_L = L.\Delta$ needs to be investigated, when evaluating therms of the Comb of Gaussians.

A Curve as a Comb

When conceived as an image, a continuous real-valued curve $(x, y) = [f(s), g(s)]$ is actually sort of a delta function. Here the arc length (s) serves as the standard running parameter.

$$C(x, y) = \delta([x - f(s)]^2 + [(y - g(s)]^2)$$

Because it is infinity (make that $= 1$ and substitute black pixels) for $(x, y) = [f(s), g(s)]$ and zero (substitute white pixels) everywhere else. In reality, of course, imaging a delta function is impossible, because:

> An ideal curve is infinitely thin
> So there is no place to put color dots in

Ideal curves are essentially invisible. However, a quite convenient *fuzzyfication* $(\overline{\delta})$ of the delta function is the Gauss function:

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2/\sigma^2} \quad \Longrightarrow \quad \overline{\delta}_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2/\sigma^2}$$

Therefore any discretized curve, at N sampling points $s = s_k$, can be made approximately continuous again, as follows (apart from a norming factor, which can always be determined afterwards).

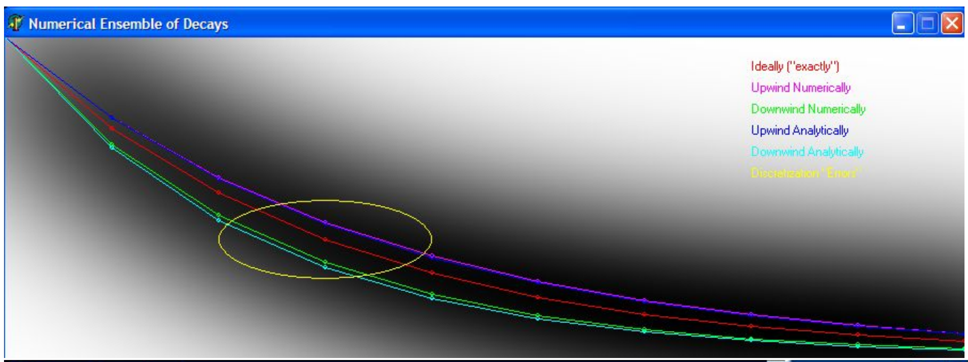
$$C(x, y) = \sum_{k=0}^{k=N} e^{-\frac{1}{2}([x-f(s_k)]^2+[y-g(s_k)]^2)/\sigma^2}$$

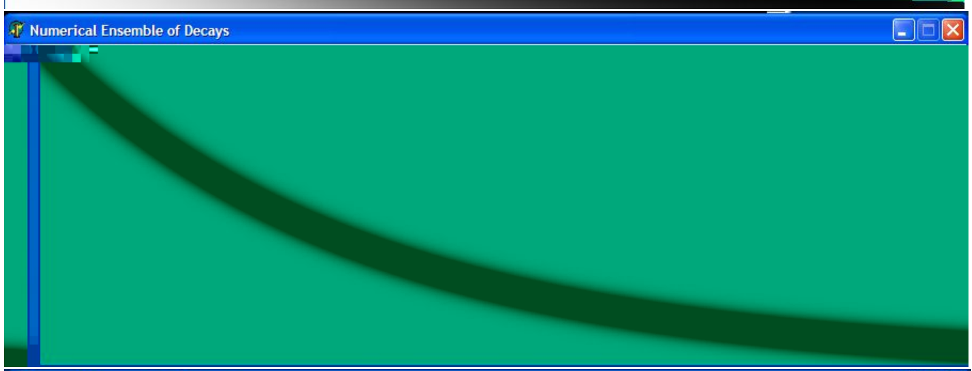
In order for the theory in the preceding subsections to be applicable, it's essential that the points s_k along the curve are more or less equidistant. And if such is not the case, define only *one* σ , which is based then upon the largest arc increment. It's left as an exercise for the reader to guess what the reason is behind these rules. Back to business now. The taste of the pudding is in the eating. You're invited to take a look at the sampled and continuized decay function / the sampled and continuized hyperbola:

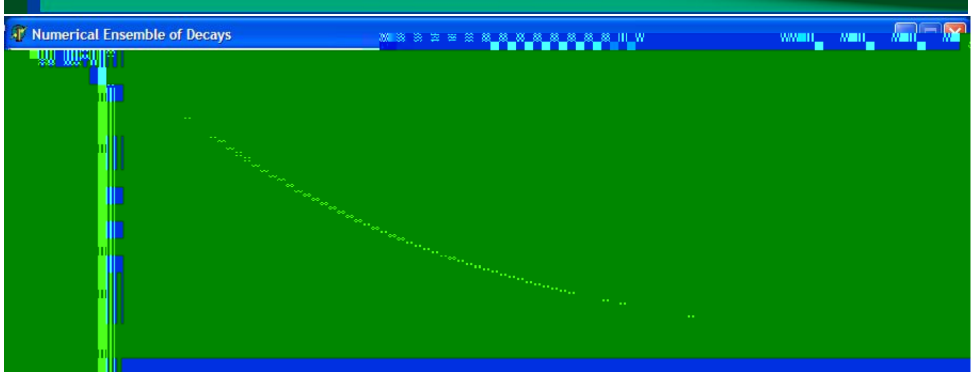
`http://hdebruijn.soo.dto.tudelft.nl/jaar2010/dikte/Project3.exe`
`http://hdebruijn.soo.dto.tudelft.nl/jaar2010/dikte/Project4.exe`

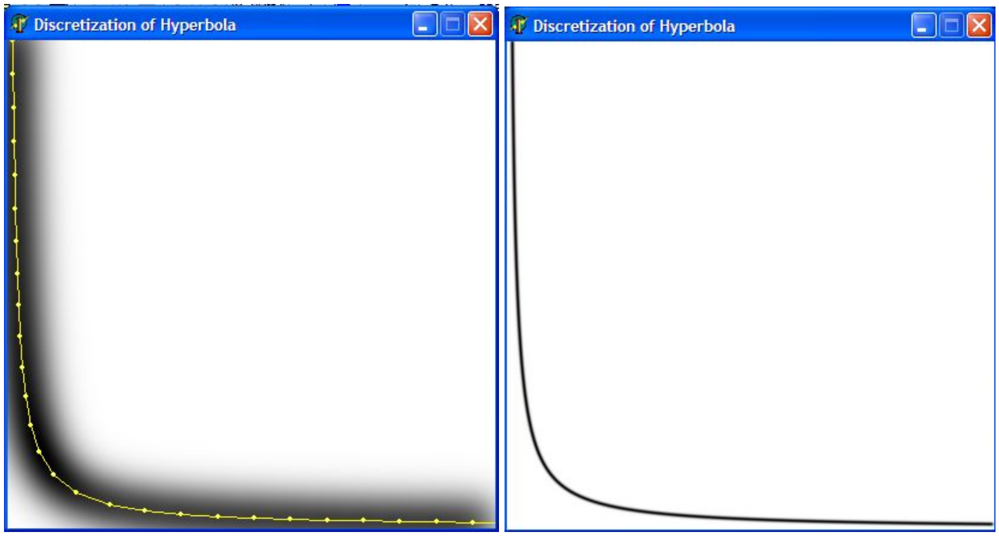
There's another story behind the sampled decay function, namely the *Numerical Ensemble of Exponential Decays*:

`http://groups.google.nl/group/sci.math/msg/d90f07f7523b0d52`









Source code is freely available as well, though still under development:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2010/dikte.zip`

A few remarks are in order. The above technique has been developed for curves in the plane. Meaning, physically, that the functions f and g in $(x, y) = [f(s), g(s)]$ have dimension of length. The spread σ has a dimension of length as well. Hence the exponent of a Gaussian, as a whole, is dimensionless. With functions instead of curves, other physical dimensions than length may be easily involved. In such cases, proper scaling of the x and y coordinates is essential. One can hardly expect consistent results if f(s) = apples and g(s) = pears. Proper scaling is accomplished most easily and naturally by the use of dimensionless quantities. With exponential decay, for example, the x coordinate is time and the y coordinate is mass; dimensionless quantities emerge if we divide time by the decay time and if we consider a mass of 1 kilogram.
With Exponential Decay, we had no other choice than leaving the uniform abscissa increments intact, though it leads to arc length increments differing at most by a factor $\sqrt{2}$ (we choose the largest one). With the hyperbola, we have employed a *predictor* for abscissa increments, given that the arc length increments ds are constant. Because it is well known that each differentiable function, within a small enough neighborhood, is a straight line segment. Thus our predictor is: $dx = ds/\sqrt{1 + (y')^2} = ds/\sqrt{1 + (1/x^2)^2}$.

Fuzzyfied Straight Line

Three steps forward, two steps backward. We are going to compare some fuzzyfied continuous entities with their discrete counterparts and see if it is possible to distinguish between the two. The two obvious candidates are a straight line and a circle. We start with the line. The equations of an ideal straight line are:

$$\begin{cases} x = a + \cos(\phi)\, t \\ y = b + \sin(\phi)\, t \end{cases}$$

Here $(x, y) =$ plane coordinates, $\phi =$ angle with horizontal (constant), $(a, b) =$ point on this line, $t =$ running parameter. All numbers real valued by default.
Lemma.

$$\begin{gathered}[x-a-\cos(\phi)\, t]^2 + [y-b-\sin(\phi)\, t]^2 \\ = [t - \{\cos(\phi)(x-a) + \sin(\phi)(y-b)\}]^2 + [\sin(\phi)(x-a) - \cos(\phi)(y-b)]^2\end{gathered}$$

Proof.

$$\begin{gathered}[x-a-\cos(\phi)\, t]^2 + [y-b-\sin(\phi)\, t]^2 \\ = (x-a)^2 + (y-b)^2 + t^2 - 2[\cos(\phi)(x-a) + \sin(\phi)(y-b)]t \\ = t^2 - 2[\cos(\phi)(x-a) + \sin(\phi)(y-b)]t + [\cos(\phi)(x-a) + \sin(\phi)(y-b)]^2 \\ -[\cos(\phi)(x-a) + \sin(\phi)(y-b)]^2 + (x-a)^2 + (y-b)^2 \\ = [t - \{\cos(\phi)(x-a) + \sin(\phi)(y-b)\}]^2 - \cos^2(\phi)(x-a)^2 - \sin^2(\phi)(y-b)^2 \\ -2\cos(\phi)(x-a)\sin(\phi)(y-b) + (x-a)^2 + (y-b)^2 \\ = [t - \{\cos(\phi)(x-a) + \sin(\phi)(y-b)\}]^2 + \sin^2(\phi)(x-a)^2 + \cos^2(\phi)(y-b)^2 \\ -2\sin(\phi)(x-a)\cos(\phi)(y-b) \\ \Longrightarrow \quad [x-a-\cos(\phi)\, t]^2 + [y-b-\sin(\phi)\, t]^2 \\ = [t - \{\cos(\phi)(x-a) + \sin(\phi)(y-b)\}]^2 + [\sin(\phi)(x-a) - \cos(\phi)(y-b)]^2\end{gathered}$$

Quite Easy Done. The (Gaussian) fuzzyfication of an ideal straight line will now be defined as follows.

$$L(x, y) = \int_{-\infty}^{+\infty} e^{-Q(x,y,t)/2} dt$$

$$\text{where} \quad Q(x, y, t) = \{[x-a-\cos(\phi)\, t]^2 + [y-b-\sin(\phi)\, t]^2\}/\sigma^2$$

With help of the lemma we find for the integral:

$$= e^{-([\sin(\phi)(x-a)-\cos(\phi)(y-b)]/\sigma)^2/2} \int_{-\infty}^{+\infty} e^{-([t-\{\cos(\phi)(x-a)+\sin(\phi)(y-b)\}]/\sigma)^2/2} dt$$

$$\Longrightarrow \quad L(x, y) = \sqrt{2\pi}\sigma\; e^{-([\sin(\phi)(x-a)-\cos(\phi)(y-b)]/\sigma)^2/2}$$

Dividing by $\sqrt{2\pi}\sigma$ results in a function with values between 0 and 1: sort of probability. Thus a *thickness* for the fuzzyfied line may be defined as being σ. Numerical experiments may be carried out now and it is for you to decide whether it's possible to see any difference between the fuzzyfied continuous and the continuized discrete:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2010/dikte/Project6.exe`

Continuing Circular

Now let's do the same for a *circle* with midpoint (a,b) and radius R:

$$C(x,y) = \sum_{k=0}^{k=N} e^{-\frac{1}{2}[A(x,y,t_k)/\sigma]^2} \qquad \text{where} \quad t_k = k.2\pi/N$$

$$\text{and} \quad A(x,y,t) = [x - a - R.\cos(t)]^2 + [y - b - R.\sin(t)]^2$$

When given the spread σ of the Gaussians, the increment Δt of the angles and hence the number N of them needed can be calculated:

$$\sigma = \frac{R\Delta t}{2\pi}\alpha = \frac{R\alpha}{N} \quad \Longrightarrow \quad N = \frac{\alpha R}{\sigma} \quad \Longrightarrow \quad \frac{R}{\sigma} = \frac{N}{\alpha}$$

There is some subtlety involved with the fact that N is an integer. But the rest is a matter of routine. When running the program ('Project2.exe'), you will notice that it's not possible to observe (i.e. see) the discretization, despite of the fact that the angle increments can be reasonably large. and thus the circle itself even more fuzzy. For comparison, the circle equation is also fuzzyfied without any discretization at all. This is the following expression.

$$C(x,y) = e^{-\frac{1}{2}[\sqrt{(x-a)^2+(y-b)^2}-R]^2/\sigma^2}$$

Contrary to alike experiments with the straight line (: previous subsection) it will be found that there *is* a difference: for "larger" angle increments.
An interesting question to ask is at which discretization the inner area of the circle becomes so much crowded with black pixels that the curve itself cannot be observed anymore. Here comes a calculation, based upon the Fourier Analysis of a preceding subsection. Assume that the Comb of Gaussians on a circle is more or less like a Comb of Gaussians on a straight line segment. Then, due to our choice of σ such that $\exp(-(2\pi/(R\Delta t)\sigma)^2/2) < \epsilon$, only the first term of that Fourier series is significant:

$$C(x,y) \approx \sigma\sqrt{2\pi}\left[\frac{1}{R\Delta t}\right]$$

This has to be compared with the contributions of the Gaussians at the midpoint of the circle:

$$Ne^{-\frac{1}{2}(R/\sigma)^2}$$

The difference of the two must be positive, in order to prevent excessive blurring:

$$\sigma\sqrt{2\pi}\frac{1}{R\Delta t} > Ne^{-\frac{1}{2}(R/\sigma)^2} \qquad \text{with:} \quad \frac{R}{\sigma} = \frac{N}{\alpha} \quad ; \quad \Delta t = \frac{2\pi}{N}$$

Simplify and express everything as a function of N :

$$\frac{\sigma}{R}\sqrt{2\pi}\frac{1}{\Delta t} = \frac{\alpha}{N}\sqrt{2\pi}\frac{N}{2\pi} > Ne^{-\frac{1}{2}(N/\alpha)^2} \quad \Longrightarrow \quad \frac{\alpha}{\sqrt{2\pi}} > Ne^{-\frac{1}{2}(N/\alpha)^2}$$

$$\Longrightarrow \quad \sqrt{2\pi}\frac{N}{\alpha}e^{-\frac{1}{2}(N/\alpha)^2} < 1 \quad \Longrightarrow \quad \frac{1}{2}\ln(2\pi) + \ln(N/\alpha) - \frac{1}{2}(N/\alpha)^2 < 0$$

When conceived as a continuous function of a real variable x it reads:

$$f(x) = \frac{1}{2}\ln(2\pi) + \ln(x) - \frac{1}{2}x^2 \qquad \text{where} \quad x = \frac{N}{\alpha} > 0$$

For $x = 1$ (hence $N = \alpha$ if such a thing were possible):

$$f(1) = \frac{1}{2}\ln(2\pi) - \frac{1}{2} > 0 \qquad \text{because} \quad 2\pi > e$$

The derivative of the function is negative for $x > 1$:

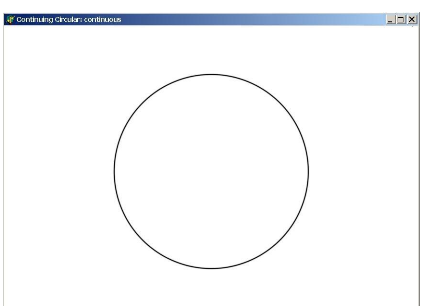
$$f'(x) = \frac{1}{x} - x = \frac{1 - x^2}{x} < 0 \qquad \text{for} \quad x > 1$$

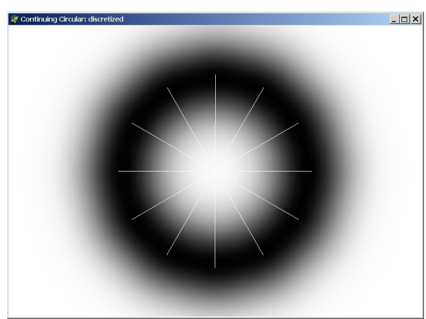
Hence the function is monotonically decreasing for $N > \alpha \approx 3.53223$, read: $N > 4$. So we can start with $N = 4$ and watch the moment that $f(x)$ becomes negative:

```
N = 4 : value =  4.02105679716663E-0001 < 0 : FALSE
N = 5 : value =  2.64575470808632E-0001 < 0 : FALSE
N = 6 : value =  6.07354288651423E-0003 < 0 : FALSE
N = 7 : value = -3.60748986496131E-0001 < 0 :  TRUE
N = 8 : value = -8.28340527575343E-0001 < 0 :  TRUE
N = 9 : value = -1.39183015011653E+0000 < 0 :  TRUE
```

From this we conclude that the situation becomes "normal" at $N = 7$. And - working the other way around - coarsening of the angle increments should not proceed beyond $N = 6$, corresponding with an angle increment of 60^o (not really a very small increment). There is software which does the slide show:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2010/dikte/Project2.exe`





At the critical angle increment $\pi/3$, our fuzzyfied circle may be considered as a hexagon. For each of the vertices of that hexagon, we calculate the contributions from itself and the other vertices, as compared with the contributions to the midpoint. Let's assume that these contributions cancel each other out:

$$1 + 2\,e^{-\frac{1}{2}(R/\sigma)^2} + 2\,e^{-\frac{1}{2}(R\sqrt{3}/\sigma)^2} + e^{-\frac{1}{2}(2R/\sigma)^2} - 6\,e^{-\frac{1}{2}(R/\sigma)^2} = 0$$

If we put:

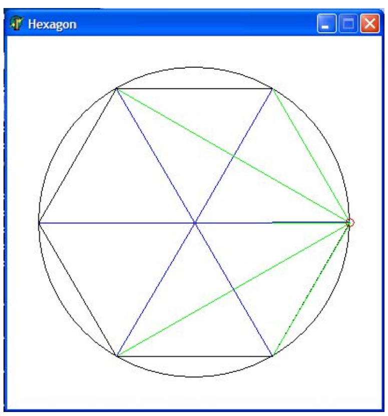
$$X = e^{-\frac{1}{2}(R/\sigma)^2} = e^{-\frac{1}{2}(\Delta/\sigma)^2} = e^{-\frac{1}{2}(2\pi/\alpha)^2} \quad \Longrightarrow \quad \alpha = \frac{2\pi}{\sqrt{2\ln(1/X)}}$$

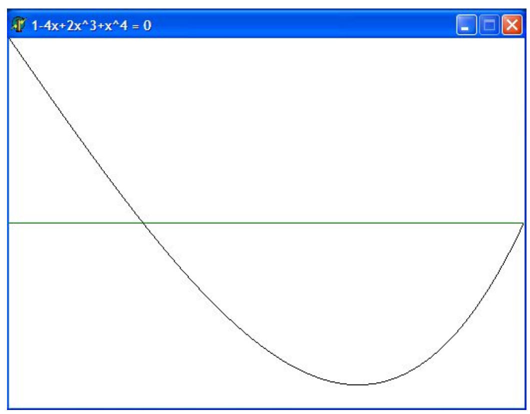
Then the above may be considered as an equation to find "break even" quantities X, $\alpha = \sqrt{2\ln(2/\epsilon)}$ and finally ϵ :

$$1 - 4X + 2X^3 + X^4 = 0 \quad \Longrightarrow \quad \alpha = \frac{2\pi}{\sqrt{2\ln(1/X)}} \quad \Longrightarrow \quad \epsilon = 2\,e^{-\frac{1}{2}\alpha^2}$$

Newton-Rhapson iteration with $X = 1/4$ as a starting value does the job. The outcomes are:

```
    X =  2.59921049894873E-0001
alpha =  3.82754450862769E+0000
  eps =  1.31765544168695E-0003 < 1/256
```





Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch.