Cone and Parabola

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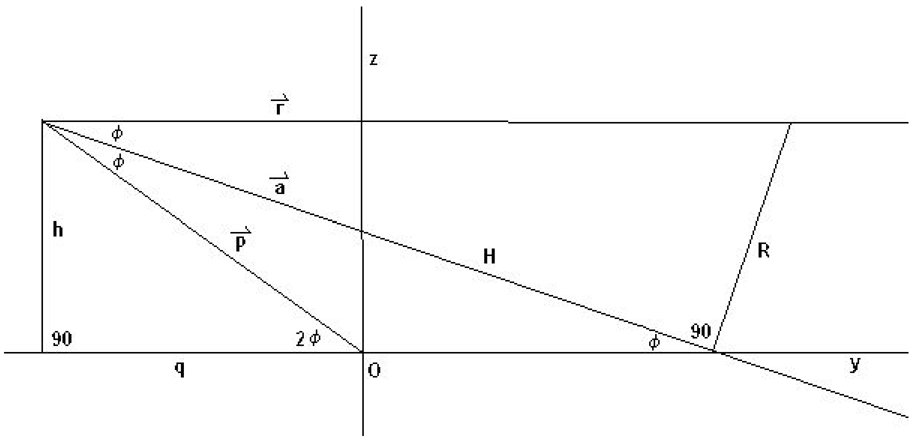

Figure 1.

Our main reference is page 3 of the following article:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2006/kegels.pdf`

The Conic Section Equation in that reference is quoted:

$$A\,x^2 + B\,xy + C\,y^2 + D\,x + E\,y + F = 0$$

Where the coefficients are given by - more quotes:

$$\begin{aligned}
A &= \cos^2(\phi) - \cos^2(\alpha)\cos^2(\gamma) \\
B &= -2\cos^2(\alpha)\cos(\gamma)\sin(\gamma) \\
C &= \cos^2(\phi) - \cos^2(\alpha)\sin^2(\gamma) \\
D &= -2A\,p - B\,q + \sin(2\alpha)\cos(\gamma)\,h \\
E &= -B\,p - 2C\,q + \sin(2\alpha)\sin(\gamma)\,h
\end{aligned}$$

And quoting an equation for F:

$$A\,p^2 + B\,pq + C\,q^2 + D\,p + E\,q + F = h^2\left[\cos^2(\phi) - \sin^2(\alpha)\right]$$

The essentials of the cone and intersecting plane configuration are projected in the (y,z) plane and depicted in figure 1. The following algebraic facts are extracted from this geometry:

$$\begin{aligned}
\gamma &= \pi/2 \\
\alpha &= -\phi \\
p &= 0 \\
\frac{h}{-q} &= \tan(2\phi)
\end{aligned}$$

Herewith the abovementioned coefficients of the Conic Section Equation are simplified considerably, because $\cos(\gamma) = 0$, $\sin(\gamma) = 1$, $\cos(\alpha) = \cos(\phi)$, $\sin(\alpha) = -\sin(\phi)$. Giving upon substitution with the rest of the data:

$$\begin{aligned} A &= \cos^2(\phi) \\ B &= 0 \\ C &= 0 \\ D &= 0 \\ E &= -\sin(2\phi)\, h \end{aligned}$$

And:

$$F = -E\, q + h^2 \left[\cos^2(\phi) - \sin^2(\phi)\right] \quad \Longrightarrow$$

$$F = \sin(2\phi)\, h\, q + h^2 \cos(2\phi) = \cos(2\phi)\, h\, q \left[\tan(2\phi) + \frac{h}{q}\right] = 0$$

The latter can also be derived from the mere fact that the origin O is on the (parabolic) curve. Whatever. Our Conic Section Equation reduces to:

$$A\, x^2 + E\, y = 0 \quad \Longrightarrow \quad \cos^2(\phi)\, x^2 - \sin(2\phi)\, h\, y = 0 \quad \Longrightarrow$$

$$y = \frac{\cos^2(\phi)\, x^2}{2h \sin(\phi) \cos(\phi)} = \frac{x^2}{2h \tan(\phi)} = \frac{x^2}{2hR/H}$$

Where R = radius of base circle and H = height of cone, as seen in figure 1. It is seen from the same figure that $h = H \sin(\phi)$ and $\sin(\phi) = R/\sqrt{H^2 + R^2}$. Upon substitution of all this we obtain:

$$y = \frac{x^2}{2H \sin(\phi) R/H} = \frac{x^2}{2R^2/\sqrt{H^2 + R^2}}$$

Which is the end result, expressing the parabola in parameters of the cone:

$$y = a\, x \qquad \text{where} \quad a = \frac{\sqrt{H^2 + R^2}}{2R^2}$$

Oh yeah, but all of the above is only valid for $a > 0$. The case $a < 0$ can be covered by mirroring the cone (see figure 1) in the (x, z) plane.

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch.