Euler-Wallis Formula for $\sin(x)$

The polynomial $z^n - 1$ posesses the roots $z = e^{i.k2\pi/n}$, $k = 0, \pm 1, \pm 2, \pm 3, \ldots$
A factorization of z^n brings us to:

$$z^n - 1 = \prod_{k=0}^{n-1} \left(z - e^{i.k.2\pi/n} \right)$$

Where z complex and k, n natural. If only odd values are allowed for n, then:

$$z^n - 1 = (z-1) \prod_{k=1}^{(n-1)/2} \left(z - e^{+i.k.2\pi/n} \right) \left(z - e^{-i.k.2\pi/n} \right)$$

$$= (z-1) \prod_{k=1}^{(n-1)/2} \left[z^2 - 2z \cos\left(\frac{k.2\pi}{n} \right) + 1 \right]$$

The geometric series is recognized in:

$$\frac{1-z^n}{1-z} = 1 + z + z^2 + z^3 + \ldots + z^{n-1}$$

For $z \to 1$ it reads:

$$\frac{1-z^n}{1-z} = 1 + 1 + 1 + 1 + \ldots + 1 = n$$

On the other hand:

$$\frac{1-z^n}{1-z} = \prod_{k=1}^{(n-1)/2} \left[z^2 - 2z \cos\left(\frac{k.2\pi}{n} \right) + 1 \right]$$

So for $z \to 1$ we infer, a result to remember:

$$n = \prod_{k=1}^{(n-1)/2} \left[2 - 2\cos\left(\frac{k.2\pi}{n} \right) \right]$$

By replacing in the general formula $z \to z/a$ and multiplying by a^n we obtain a slightly more general result:

$$z^n - a^n = (z-a) \prod_{k=1}^{(n-1)/2} \left[z^2 - 2za \cos\left(\frac{k.2\pi}{n} \right) + a^2 \right]$$

We now insert $z \to (1 + z/n)$ and $a \to (1 - z/n)$. This gives:

$$(1+z/n)^n - (1-z/n)^n = \frac{2z}{n} \prod_{k=1}^{(n-1)/2} \left[2\left(1 + \frac{z^2}{n^2}\right) - 2\left(1 - \frac{z^2}{n^2}\right) \cos\left(\frac{k.2\pi}{n} \right) \right]$$

$$= \frac{2z}{n} \prod_{k=1}^{(n-1)/2} \left[\left\{ 2 - 2\cos\left(\frac{k.2\pi}{n}\right) \right\} + \frac{z^2}{n^2} \left\{ 2 + 2\cos\left(\frac{k.2\pi}{n}\right) \right\} \right]$$

$$= \frac{2z}{n} \prod_{k=1}^{(n-1)/2} \left[2 - 2\cos\left(\frac{k.2\pi}{n}\right) \right] \left[1 + \left(\frac{z^2}{n^2}\right) \frac{1 + \cos(k.2\pi/n)}{1 - \cos(k.2\pi/n)} \right]$$

$$= \frac{2z}{n} \prod_{k=1}^{(n-1)/2} \left[2 - 2\cos\left(\frac{k.2\pi}{n}\right) \right] \prod_{k=1}^{(n-1)/2} \left[1 + \left(\frac{z^2}{n^2}\right) \frac{1 + \cos(k.2\pi/n)}{1 - \cos(k.2\pi/n)} \right]$$

With the result you're supposed to remember:

$$(1 + z/n)^n - (1 - z/n)^n = 2z \prod_{k=1}^{(n-1)/2} \left[1 + \left(\frac{z^2}{n^2}\right) \frac{1 + \cos(k.2\pi/n)}{1 - \cos(k.2\pi/n)} \right]$$

The factor $[1 + \cos(k.2\pi/n)] \,/\, [1 - \cos(k.2\pi/n)]$ shall be simplified with help of well-known trigonometric formulas for the cosine of a double angle:

$$\frac{1 + \cos(k.2\pi/n)}{1 - \cos(k.2\pi/n)} = \frac{2\cos^2(k.\pi/n)}{2\sin^2(k.\pi/n)} = \frac{1}{\tan^2(k.\pi/n)}$$

Resulting in:

$$\frac{(1 + z/n)^n - (1 - z/n)^n}{2} = z \prod_{k=1}^{(n-1)/2} \left[1 + \left\{ \frac{z}{n.\tan(k.\pi/n)} \right\}^2 \right]$$

The tangent only assumes values here for $0 < k.\pi/n < \pi/2$. Therefore define the following monotonic function T on the positive reals:

$$T(x) = \begin{cases} 1 & : \quad x \le 0 \\ x/\tan(x) & : \quad 0 < x < \pi/2 \\ 0 & : \quad x \ge \pi/2 \end{cases}$$

Giving that the rightmost product in the following expression will be equal to one, because $k.\pi/n > \pi/2$ and hence $T(k.\pi/n) = 0$ at that place. This allows for writing as an infinite product:

$$\frac{(1 + z/n)^n - (1 - z/n)^n}{2} = z \prod_{k=1}^{\infty} \left[1 + \left\{ \frac{z}{k.\pi} T(k.\pi/n) \right\}^2 \right] =$$

$$z \prod_{k=1}^{(n-1)/2} \left[1 + \left\{ \frac{z}{k.\pi} T(k.\pi/n) \right\}^2 \right] \prod_{k=(n+1)/2}^{\infty} \left[1 + \left\{ \frac{z}{k.\pi} T(k.\pi/n) \right\}^2 \right]$$

Now it's just a calculus exercise to show that:

$$\lim_{n\to\infty} T(k.\pi/n) = \lim_{n\to\infty} \frac{k.\pi/n}{\tan(k.\pi/n)} = \lim_{\Delta\to 0} \left[\frac{\Delta}{\sin(\Delta)} \right] \cos(\Delta) = 1$$

The limit of the product is the product of the limits:

$$\frac{e^z - e^{-z}}{2} = \lim_{n\to\infty} z \prod_{k=1}^{\infty} \left(1 + \left\{ \frac{z}{\pi k} \, T(k.\pi/n) \right\}^2 \right) = \prod_{k=1}^{\infty} \left(1 + \left\{ \frac{z}{\pi k} .1 \right\}^2 \right)$$

Replacing $z \to i.\pi.z$, we finally obtain the Euler-Wallis formula:

$$\sin(\pi.z) = \pi.z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2} \right)$$

Acknowledgements

Much of this article is simply a cut and paste from (public as well as private) contributions by Rob Johnson, David C. Ullrich, Rob Johnson, and a person calling him/herself PM. See:

`http://groups.google.com/group/sci.math/browse_thread/thread/b5125eb7ce990fad/`
`http://www.mathlinks.ro/Forum/viewtopic.php?t=91010  : message #4`
`http://www.whim.org/nebula/math/pdf/eulerwallis.pdf`

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