Cauchy-Riemann Equations

Author: Han de Bruijn
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It is shown in this article that a complex function $f = u + i\,v$ is complex differentiable to $z = x + i\,y$ if and only if it is real differentiable at z and if the partial derivatives of u and v to x and y obey the so called Cauchy-Riemann Equations: $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$.

Two special cases

The complex derivative $f'(z)$ of a complex function $f(z)$ with z complex is defined as follows (with Δz complex as well).

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

A complex function can always be written in real valued components (x, y, u, v) as:

$$f(z) = u(x, y) + i.v(x, y) \qquad \text{where} \quad z = x + i.y$$

Complex differentiation is independent of the direction, the way z approaches zero. Two special cases are distinguished: differentiation in the x-direction and differentiation in the y-direction. In the x-direction it is found that $\Delta z = \Delta x$ and the complex derivative is:

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y) + i\,[v(x + \Delta x, y) - v(x, y)]}{\Delta x} = \frac{\partial u}{\partial x} + i.\frac{\partial v}{\partial x}$$

In the y-direction it is found that $\Delta z = i.\Delta y$ and the complex derivative is:

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y) + i\,[v(x, y + \Delta y) - v(x, y)]}{i.\Delta y} = -i.\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Two complex numbers are equal if and only if the real and the imaginary parts are equal:

$$f'(z) = \frac{\partial u}{\partial x} + i.\frac{\partial v}{\partial x} = -i.\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \Longrightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These are the well-known Cauchy-Riemann equations. Conclusion: if a function $f = u + i.v$ is complex differentiable, then the real and imaginary parts u and v of f satisfy the Cauchy-Riemann equations at $z = x + i.y$.

Real Differentiable

The complex derivative $f'(z)$ of a complex function $f(z)$ with z complex is defined as follows (with Δz complex as well).

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

Consequently:

$$f(z + \Delta z) = f(z) + f'(z)\Delta z \qquad \text{for} \quad z \to 0$$

A complex function can always be written in real valued components (x, y, u, v) as:

$$f(z) = u(x, y) + i.v(x, y) \qquad \text{where} \quad z = x + i.y$$

Consequently, for $\Delta x \to 0$ and $\Delta y \to 0$:

$$u(x + \Delta x, y + \Delta y) + i.v(x + \Delta x, y + \Delta y) = u(x, y) + i.v(x, y)$$
$$+ [u'(x, y) + i.v'(x, y)] [\Delta x + i\Delta y] \quad \Longrightarrow$$
$$u(x + \Delta x, y + \Delta y) = u(x, y) + u'(x, y)\Delta x - v'(x, y)\Delta y$$
$$v(x + \Delta x, y + \Delta y) = v(x, y) + v'(x, y)\Delta x + u'(x, y)\Delta y$$

Meaning that, if a function $f = u + i.v$ is complex differentiable at $z = x + i.y$, then its real and imaginary parts (u, v) are real differentiable at (x, y) . On the other hand it is known from real analysis that:

$$u(x + \Delta x, y + \Delta y) = u(x, y) + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y \qquad \text{for} \quad (\Delta x, \Delta y) \to (0, 0)$$

$$v(x + \Delta x, y + \Delta y) = v(x, y) + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y \qquad \text{for} \quad (\Delta x, \Delta y) \to (0, 0)$$

This can only be consistent if the Cauchy-Riemann equations are indeed valid:

$$\frac{\partial u}{\partial x} = u'(x, y) = \frac{\partial v}{\partial y} \qquad ; \qquad \frac{\partial v}{\partial x} = v'(x, y) = -\frac{\partial u}{\partial y}$$

It is known from real analysis that a function $f(x, y)$ is total differentiable at (a, b) , if and only if the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist in a neighbourhood of (a, b) and both are continuous there. Thus making the picture complete.

Independent of angle

The reverse question is: if the Cauchy-Riemann equations hold for the real and imaginary parts of a complex function f, is f complex differentiable then? To answer this question, let's write the complex derivative with $\Delta z = r.e^{i\theta}$. The reason is that the complex derivative must be independent of any (real) angle θ while the (real) distance r from $z + \Delta z$ to z approaches zero. So this is what we do:

$$f'(z) = \lim_{r \to 0} \frac{f(z + r.e^{i\theta}) - f(z)}{r.e^{i\theta}}$$

Remember that $f(z) = u(x, y) + i.v(x, y)$ where $z = x + i.y$. Also remember the jewel formula by Euler $e^{i\theta} = \cos(\theta) + i.\sin(\theta)$. Giving:

$$f'(z) = \lim_{r \to 0} \frac{u(x + r\cos(\theta), y + r\sin(\theta)) - u(x, y)}{r\cos(\theta) + i.r\sin(\theta)}$$

$$+\,i\,\lim_{r\to 0}\frac{v(x+r\cos(\theta),y+r\sin(\theta))-v(x,y)}{r\cos(\theta)+i.r\sin(\theta)}$$

Given a real differentiable function $g(x,y)$, for $(h,k)\to 0$, by definition (of just being real differentiable):

$$g(x+h,y+k)-g(x,y)=\frac{\partial g}{\partial x}(x,y)\,h+\frac{\partial g}{\partial y}(x,y)\,k$$

With $g=(u,v)$ and $[h,k]=[r\cos(\theta),r\sin(\theta)]$:

$$f'(z)=\lim_{r\to 0}\frac{\partial u/\partial x.r\cos(\theta)+\partial u/\partial y.r\sin(\theta)}{r\cos(\theta)+i.r\sin(\theta)}$$

$$+\,i\,\lim_{r\to 0}\frac{\partial v/\partial x.r\cos(\theta)+\partial v/\partial y.r\sin(\theta)}{r\cos(\theta)+i.r\sin(\theta)}$$

Actually taking the limit for $r\to 0$ is easy:

$$f'(z)=\frac{\partial u/\partial x.\cos(\theta)+\partial u/\partial y.\sin(\theta)}{\cos(\theta)+i\sin(\theta)}\;+\;i\;\frac{\partial v/\partial x.\cos(\theta)+\partial v/\partial y.\sin(\theta)}{\cos(\theta)+i\sin(\theta)}$$

Substitute herein the Cauchy-Riemann equations (by a copy and paste from the preceding subsection):

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\quad\text{and}\quad\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

Just doit:

$$f'(z)=\frac{\partial u/\partial x.\cos(\theta)-\partial v/\partial x.\sin(\theta)}{\cos(\theta)+i\sin(\theta)}\;+\;i\;\frac{\partial v/\partial x.\cos(\theta)+\partial u/\partial x.\sin(\theta)}{\cos(\theta)+i\sin(\theta)}$$

$$f'(z)=\frac{\partial u}{\partial x}\,\frac{\cos(\theta)+i\sin(\theta)}{\cos(\theta)+i\sin(\theta)}\;+i\;\frac{\partial v}{\partial x}\,\frac{\cos(\theta)+i\sin(\theta)}{\cos(\theta)+i\sin(\theta)}=\frac{\partial u}{\partial x}+i\,\frac{\partial v}{\partial x}$$

Which indeed is independent of any angle θ . Therefore, if the function $f=u+i.v$ is real differentiable at $z=x+i.y$ and if the partial derivatives of u and v to x and y obey the Cauchy-Riemann Equations, then the complex derivative of f is independent of the direction in which we differentiate. But this is precisely what we *mean* by the phrase "complex differentiable".

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