Counting Zeroes of Polynomials

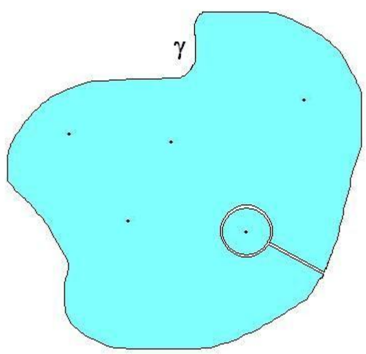
According to the Fundamental Theorem of Algebra, any complex polynomial of degree n can be written as:

$$P_n(z) = a_n \prod_{j=1}^{n} (z - z_j)$$

Where z_i are the n zeroes of the polynomial. Now, if we could only write:

$$\frac{1}{2\pi.i} \int_\gamma \left[\sum_{j=1}^{n} \frac{1}{z - z_j} \right] dz = \sum_{j=1}^{n} \frac{1}{2\pi.i} \int_\gamma \frac{dz}{z - z_j}$$

Where $\gamma(t)$ is a simple closed curve. Then the result of the integration would be a sum of well known integrals, each to be evaluated in a geometry like depicted below. That is: with a circle $o(t) = z_j + r.e^{i.t}$ around each of the zeroes.



$$\frac{1}{2\pi.i} \int_\gamma \left[\sum_{j=1}^{n} \frac{1}{z - z_j} \right] dz = \sum_{j=1}^{n} \frac{1}{2\pi.i} \oint \frac{dz}{z - z_j} = \sum_{j=1}^{n} 1 = n$$

Can we do this? Yes we can:

$$\sum_{j=1}^{n} \frac{1}{z - z_j} =$$

$$\frac{1}{z - z_1} + \frac{1}{z - z_2} + \ldots + \frac{1}{z - z_j} + \ldots + \frac{1}{z - z_n} =$$

$$\frac{(z - z_2)(z - z_3)\ldots(z - z_j)\ldots(z - z_n)}{(z - z_1)(z - z_2)(z - z_3)\ldots(z - z_j)\ldots(z - z_n)} +$$

$$\frac{(z - z_1)(z - z_3)\ldots(z - z_j)\ldots(z - z_n)}{(z - z_1)(z - z_2)(z - z_3)\ldots(z - z_j)\ldots(z - z_n)} + \ldots +$$

$$\frac{(z-z_1)(z-z_2)\dots(z-z_{j-1})(z-z_{j+1})\dots(z-z_n)}{(z-z_1)(z-z_2)(z-z_3)\dots(z-z_j)\dots(z-z_n)} + \dots +$$

$$\frac{(z-z_2)(z-z_3)\dots(z-z_j)\dots(z-z_{n-1})}{(z-z_1)(z-z_2)(z-z_3)\dots(z-z_j)\dots(z-z_n)}$$

So, with the product rule for differentiation:

$$\sum_{j=1}^{n} \frac{1}{z-z_j} = \frac{\frac{d}{dz}\prod_{j=1}^{n}(z-z_j)}{\prod_{j=1}^{n}(z-z_j)} = \frac{P_n'(z)}{P_n(z)}$$

Conclusion:

$$n = \frac{1}{2\pi.i}\int_\gamma \left[\sum_{j=1}^{n}\frac{1}{z-z_j}\right] dz = \frac{1}{2\pi.i}\int_\gamma \frac{P_n'(z)}{P_n(z)}dz \quad ; \quad P_n(z) = a_n \prod_{j=1}^{n}(z-z_j)$$

Which is a special case of "Counting Zeroes" in Chapter 5 of Ullrich's book, but obtained with knowledge no more advanced than in the middle of Chapter 3. The important thing is that we don't have to know the zeroes in order to be able to *count* the zeroes. Because if we know the polynomial $P_n(z)$ then we only have to differentiate it to obtain $P_n'(z)$.