Least Squares Means

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The weighted mean of two one-dimensional distances between one unknown and two constants is:

$$\begin{gathered}
w_1(x-A)^2 + w_2(x-B)^2 = \\
w_1x^2 - 2w_1Ax + w_1A^2 + w_2x^2 - 2w_2Bx + w_2B^2 = \\
(w_1+w_2)\left[x^2 - 2\frac{w_1A+w_2B}{w_1+w_2}x + \left(\frac{w_1A+w_2B}{w_1+w_2}\right)^2\right] \\
-\frac{(w_1A+w_2B)^2}{w_1+w_2} + \frac{(w_1A^2+w_2B^2)(w_1+w_2)}{w_1+w_2} = \\
(w_1+w_2)\left[x - \frac{w_1A+w_2B}{w_1+w_2}\right]^2 \\
-\frac{w_1^2A^2 + 2w_1w_2AB + w_2^2B^2}{w_1+w_2} + \frac{w_1^2A^2 + w_1w_2A^2 + w_2^2B^2 + w_1w_2B^2}{w_1+w_2} = \\
(w_1+w_2)\left[x - \frac{w_1A+w_2B}{w_1+w_2}\right]^2 + \frac{w_1w_2}{w_1+w_2}(A^2 - 2AB + B^2)
\end{gathered}$$

The result is a *Lemma*:

$$w_1(x-A)^2 + w_2(x-B)^2 = (w_1+w_2)\left[x - \frac{w_1A+w_2B}{w_1+w_2}\right]^2 + \frac{w_1w_2}{w_1+w_2}(A-B)^2$$

Which is the sum of two squares. But, unless $A = B$, the second square cannot be zero. Therefore it is called the residual. Only the first square can be zero by putting the unknown equal to the weighted mean of the two constants.
This is a general phenomenon with least squares methods. For example, let:

$$\sum_i w_i(x-A_i)^2 = \text{minimum}(x)$$

Then by differentiation to x :

$$2\sum_i w_i(x-A_i) = 0 \quad \Longrightarrow \quad x = \frac{\sum_i w_iA_i}{\sum_i w_i}$$

Thus minimizing the distances between an unknown value and a set of constants by means a weighted least squares method results, again, in the unknown being equal to the arithmetic mean of the constants.

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch :-)