Derivatives of Carriers

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Gaussian Carrier

One of the distance carrier functions, employed with my 'Sunk into an Idea' project, is one minus the Gauss function. In one dimension, this carrier function would read as:

$$P(|x_i - x_m|) = 1 - e^{-(x_i - x_m)^2/2\sigma^2}$$

Where (i) is the index for the idea and (m) is the index for the material. With only one point in the idea, the arithmetic mean of the geometric mean being a minimum would read as:

$$P(x) = \sum_{m=0}^{M-1} \left[1 - e^{-(x-x_m)^2/2\sigma^2}\right] = \text{minimum}(x)$$

Where the index of the idea has been removed, because from now on it will be considered as an independent variable. The carrier function $P(x)$ shall be differentiated with respect to x and put equal to zero:

$$P'(x) = \sum_{m=0}^{M-1} \frac{x - x_m}{\ \ ^2} \left[1 - e^{-(x-x_m)^2/2\sigma^2}\right] = 0$$

Unfortunately, this equation is nonlinear. So Newton-Rhapson iterations come into mind. To get finished with these, the second derivative is needed:

$$P''(x) = \sum_{m=0}^{M-1} \left[-\left(\frac{x - x_m}{\ \ ^2}\right)^2 + \frac{1}{\ \ ^2}\right] \left[1 - e^{-(x-x_m)^2/2\sigma^2}\right]$$

With Newton-Rhapson iterations, the second derivative should be positive. This is certainly the case (sufficient condition) if:

$$-\left(\frac{x - x_m}{\ \ ^2}\right)^2 + \frac{1}{\ \ ^2} > 0 \quad \Longrightarrow \quad |x - x_m| <$$

In words: with a Gaussian distribution as a carrier, the idea must be somewhere in the neighbourhood of the material. A crude estimate for the spread of this neigbourhood is   , the spread of the Gaussian distribution.

Cauchy Carrier

One of the distance carrier functions, employed with my 'Sunk into an Idea' project, is one minus the Cauchy distribution function. In one dimension, this carrier function would read as:

$$P(|x_i - x_m|) = \frac{(x_i - x_m)^2}{(x_i - x_m)^2 + \ \ ^2}$$

Where (i) is the index for the idea and (m) is the index for the material. With only one point in the idea, the arithmetic mean of the geometric mean being a minimum would read as:

$$P(x) = \sum_{m=0}^{M-1} \left[\frac{(x - x_m)^2}{(x - x_m)^2 + \quad {}^2} \right] = \text{minimum}(x)$$

Where the index of the idea has been removed, because from now on it will be considered as an independent variable. The carrier function $P(x)$ shall be differentiated with respect to x and put equal to zero:

$$P'(x) = \sum_{m=0}^{M-1} \frac{2(x - x_m)\left[(x - x_m)^2 + \quad {}^2\right] - (x - x_m)^2\, 2(x - x_m)}{[(x - x_m)^2 + \quad {}^2]^2}$$

$$= \sum_{m=0}^{M-1} \frac{2(x - x_m) \quad {}^2}{[(x - x_m)^2 + \quad {}^2]^2} = 0$$

Unfortunately, this equation is nonlinear. So Newton-Rhapson iterations come into mind. To get finished with these, the second derivative is needed:

$$P''(x) = \sum_{m=0}^{M-1} \frac{2 \quad {}^2 \left[(x - x_m)^2 + \quad {}^2\right]^2 - 4(x - x_m)\left[(x - x_m)^2 + \quad {}^2\right] :2(x - x_m) \quad {}^2}{[(x - x_m)^2 + \quad {}^2]^4}$$

$$= \sum_{m=0}^{M-1} \frac{2 \quad {}^2 \left[(x - x_m)^2 + \quad {}^2\right] - 8(x - x_m)^2 \quad {}^2}{[(x - x_m)^2 + \quad {}^2]^3} = 0$$

With Newton-Rhapson iterations, the second derivative should be positive. This is certainly the case (sufficient condition) if:

$$-6(x - x_m)^2 \quad {}^2 + 2 \quad {}^4 > 0 \quad \Longrightarrow \quad (x - x_m)^2 < \quad {}^2{=}3 \quad \Longrightarrow \quad |x - x_m| < \frac{\quad}{\sqrt{3}}$$

In words: with a Cauchy distribution as a carrier, the idea must be somewhere in the neighbourhood of the material. A crude estimate for the spread of this neigbourhood is $\quad = \sqrt{3}$, where   is the spread of the Cauchy distribution.

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch :-)