Equality and Logic Fuzzyfied

Author: Han de Bruijn
Date: 2006 June

This paper is concerned with the materialization of the most basic notion in the whole of mathematics: **equality**. Two approaches are presented. The first one is a fuzzyfication of $(a = b)$ with help of Gaussian distributions. The second one is a generalization of boolean algebra for quantities which can assume any value *between* 0 and 1, instead of just 0 or 1. Both approaches will be employed as a technology to find solution for a well known children's puzzle: find the differences between two "equal" pictures.

Renormalization of Equality

There are situations in practice where an equality less than "exact" is quite desirable. The present paper aims to provide such an equality.

Definition.

Given an uncertainity σ, a number a is equal to b for P percent if and only if

$$e^{-\frac{1}{2}(a-b)^2/\sigma^2} = P$$

Solving for $|a - b|$:

$$-\frac{1}{2}(a-b)^2/\sigma^2 = \ln(P) \quad \Longrightarrow \quad (a-b)^2/\sigma^2 = -2\ln(P) = \ln(1/P^2) \quad \Longrightarrow$$

$$|a - b| = \sigma\sqrt{\ln(1/P^2)}$$

Examples:

For $|a - b| = \sigma$: $\sqrt{\ln(1/P^2)} = 1 \quad \Longrightarrow \quad P = 1/\sqrt{e} \approx 60.7$ percent.

For $|a - b| = \frac{1}{2}\sigma$: $\sqrt{\ln(1/P^2)} = 1/2 \quad \Longrightarrow \quad P = 1/\sqrt{\sqrt{\sqrt{e}}} \approx 88.2$ percent.

Theorem (Reflexivity).

For all uncertainities, any number is 100 percent equal to itself.

Proof.

$$e^{-\frac{1}{2}(a-a)^2/\sigma^2} = 1$$

Theorem (Symmetry).

If a is P percent equal to b, then b is P percent equal to a.

Proof.

$$e^{-\frac{1}{2}(a-b)^2/\sigma^2} = P = e^{-\frac{1}{2}(b-a)^2/\sigma^2}$$

Lemma.

$$w_1(x-A)^2 + w_2(x-B)^2 =$$

$$(w_1 + w_2)\left(x - \frac{w_1 A + w_2 B}{w_1 + w_2}\right)^2 + \frac{w_1 w_2}{w_1 + w_2}(A - B)^2$$

Proof.

$$w_1(x-A)^2 + w_2(x-B)^2 = w_1x^2 - 2w_1xA + w_1A^2 + w_2x^2 - 2w_2xB + w_2B^2 =$$

$$(w_1+w_2)\left[x^2 - 2\frac{w_1A+w_2B}{w_1+w_2}x + \left(\frac{w_1A+w_2B}{w_1+w_2}\right)^2\right]$$

$$-\frac{(w_1A+w_2B)^2}{w_1+w_2} + w_1A^2 + w_2B^2 =$$

$$(w_1+w_2)\left[x - \frac{w_1A+w_2B}{w_1+w_2}\right]^2$$

$$+\frac{-w_1^2A^2 - 2w_1w_2AB - w_2^2B^2}{w_1+w_2} + \frac{w_1^2A^2 + w_1w_2A^2 + w_1w_2B^2 + w_2^2B^2}{w_1+w_2} =$$

$$(w_1+w_2)\left(x - \frac{w_1A+w_2B}{w_1+w_2}\right)^2 + \frac{w_1w_2}{w_1+w_2}(A-B)^2$$

Theorem (Transitivity).
If a is P_1 percent equal to b with uncertainity σ_1, and b is P_2 percent equal to c with uncertainity σ_2, then a is equal to c with a percentage greater than P_1P_2 and an uncertainity $\sqrt{\sigma_1^2+\sigma_2^2}$.
Proof.

$$e^{-\frac{1}{2}(a-b)^2/\sigma^2}e^{-\frac{1}{2}(b-c)^2/\sigma^2} = P_1P_2 = e^{-\frac{1}{2}\left[w_1(x-A)^2+w_2(x-B)^2\right]} =$$

$$e^{-\frac{1}{2}(w_1+w_2)\left(x-\frac{w_1A+w_2B}{w_1+w_2}\right)^2}e^{-\frac{1}{2}\frac{w_1w_2}{w_1+w_2}(A-B)^2} = P_1P_2$$

Where $x=b$, $A=a$, $B=c$, $w_1=1/\sigma_1^2$, $w_2=1/\sigma_2^2$ in the above Lemma. Hence:

$$e^{-\frac{1}{2}\frac{w_1w_2}{w_1+w_2}(A-B)^2} = P_1P_2e^{+\frac{1}{2}(w_1+w_2)\left(x-\frac{w_1A+w_2B}{w_1+w_2}\right)^2}$$

Where the latter exponential function may be expected to be close to 1 but anyway is greater than 1. Furthermore it is seen that

$$\frac{w_1w_2}{w_1+w_2} = \frac{1/\sigma_1^2.1/\sigma_2^2}{1/\sigma_1^2+1/\sigma_2^2} = \frac{1}{\sigma_1^2+\sigma_2^2}$$

And last but not least $(A-B)^2=(a-c)^2$. Thus:

$$e^{-\frac{1}{2}(a-c)^2/(\sigma_1^2+\sigma_2^2)} > P_1P_2$$

Which proves the theorem on Transitivity.

Fuzzyfied Logic

Let x, y, z sets, propositions or predicates. Then for these elements the axioms of **boolean algebra** are supposed to be valid:

$$
\begin{aligned}
z \cap z &= z \\
z \cup z &= z \\
y \cap z &= z \cap y \\
y \cup z &= z \cup y \\
x \cap (y \cap z) &= (x \cap y) \cap z \\
x \cup (y \cup z) &= (x \cup y) \cup z \\
x \cap (y \cup z) &= (x \cap y) \cup (x \cap z) \\
x \cup (y \cap z) &= (x \cup y) \cap (x \cup z) \\
0 \cap z &= 0 \\
1 \cup z &= 1 \\
1 \cap z &= z \\
0 \cup z &= z \\
\overline{y \cap z} &= \overline{y} \cup \overline{z} \\
\overline{y \cup z} &= \overline{y} \cap \overline{z} \\
z \cap \overline{z} &= 0 \\
z \cup \overline{z} &= 1 \\
\overline{\overline{z}} &= z
\end{aligned}
$$

The above axioms are valid for numbers x, y, z which assume values from the set $\{0, 1\}$. W

$$\begin{aligned}
\overline{y \cap z} &= \overline{y} \cup \overline{z} \\
\overline{y \cup z} &= \overline{y} \cap \overline{z} \\
\overline{\overline{z}} &= z
\end{aligned}$$

Hints:

$$\begin{aligned}
x \cup (y \cup z) &= x + y + z - x.y - x.z - y.z + x.y.z \\
\overline{y \cup z} &= 1 - (y + z - y.z) \\
\overline{y} \cap \overline{z} &= (1-y)(1-z) \\
\overline{y \cap z} &= 1 - y.z \\
\overline{y} \cup \overline{z} &= (1-y) + (1-z) - (1-y).(1-z)
\end{aligned}$$

But the other axioms are **no** longer valid:

$$\begin{aligned}
z \cap z &\neq z \\
z \cup z &\neq z \\
x \cap (y \cup z) &\neq (x \cap y) \cup (x \cap z) \\
x \cup (y \cap z) &\neq (x \cup y) \cap (x \cup z) \\
z \cap \overline{z} &\neq 0 \\
z \cup \overline{z} &\neq 1
\end{aligned}$$

Hints:

$$\begin{aligned}
x \cap (y \cup z) &= x.(y + z - y.z) \\
(x \cap y) \cup (x \cap z) &= x.y + x.z - (x.y)(x.z) \\
x \cup (y \cap z) &= x + y.z - x.y.z \\
(x \cup y) \cap (x \cup z) &= (x + y - x.y).(x + z - x.z)
\end{aligned}$$

The following truth table (: logic) and Karnaugh diagram (: set theory) are valid for equivalent (read: **equal**) boolean elements.

```
 a | b | a = b                ---------
---------------            a |    | XX |
 0 | 0 |   1                  ---------
 0 | 1 |   0                 | XX |    |
 1 | 0 |   0                  ---------
 1 | 1 |   1                      b
```

Thus it is seen that fuzzy equality corresponds with the following non-trivial arithmetical expression.

$$(a \cap b) \cup (\overline{a} \cap \overline{b}) = (a.b) \cup (1-a)(1-b) =$$

$$a.b + (1-a)(1-b) - a.b(1-a)(1-b) = a.b + (1-a)(1-b)(1-a.b) =$$
$$1 - a - b + a.b + a^2.b + a.b^2 - a^2.b^2$$

Find the Differences

A application of Fuzzy Equal as well as Fuzzy Logic is the detection of differences between almost equal pictures:

> `http://hdebruijn.soo.dto.tudelft.nl/jaar2006/MARGRIET.ZIP`
> `http://hdebruijn.soo.dto.tudelft.nl/jaar2006/PLAATJES.ZIP`

When compared to Fuzzy Equality, the Fuzzy Logic approach seems to be the better of the two.
But there is more to it. In classical logic the intersection between $(a \cap b)$ and $(\overline{a} \cap \overline{b})$ is zero: the sets $(a \cap b)$ and $(\overline{a} \cap \overline{b})$ are disjoint. Thus $a.b.(1-a)(1-b) = 0$. This means that a suitable fuzzyfication could also be given by:

$$(a \cap b) \cup (\overline{a} \cap \overline{b}) = a.b + (1-a)(1-b)$$

That is: just a bilinear interpolation between $[(0,0),(1,1)\text{with equal} = 1]$ and $[(1,0),(0,1)\text{with not equal} = 0]$. It can be shown that this approximation works equally well as the original one.
Even worse. I have been told (by Robert Low in 'sci.math') that there exists a 'standard' version of Fuzzy Logic, as can be found at:

> `http://en.wikipedia.org/wiki/Fuzzy_set_operations`

Union, intersection and complement are defined in this 'standard' version as:

$$\begin{aligned} a \cap b &:= \min(a,b) \\ a \cup b &:= \max(a,b) \\ \overline{a} &:= 1-a \end{aligned}$$

And - surprise, surprise - this version works equally well as the other two ! The main conclusion herefrom is that the fuzzyfication of boolean logic is *not quite unambiguous*. This is a common feature with materializations, as is exemplified numerous times with e.g. Numerical Analysis: there do exist many numerical approximations for the same differential equation.

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch.