Sensible Directions

Author: Han de Bruijn
Date: 2006 July

The self energy of a Gaussian Potential Field is employed for the purpose of finding places in a picture where straight line segments are present. Necessary conditions are that the gradient of the field is zero and that the determinant of the Jacobian matrix is positive at those places.

Gaussian Potential Field

The Gaussian Potential Field is defined as:

$$V(x,y) = -\sum_k e^{-\frac{1}{2}\left[(x-x_k)^2+(y-y_k)^2\right]/\sigma^2}$$

Where (x,y) = coordinates, k = pixel point, σ = spread. Partial derivatives:

$$\sigma\frac{\partial V}{\partial x} = +\sum_k \frac{x-x_k}{\sigma} e^{-\frac{1}{2}\left[(x-x_k)^2+(y-y_k)^2\right]/\sigma^2}$$

$$\sigma\frac{\partial V}{\partial y} = +\sum_k \frac{y-y_k}{\sigma} e^{-\frac{1}{2}\left[(x-x_k)^2+(y-y_k)^2\right]/\sigma^2}$$

The gradient of the Gaussian Potential Field acts like a *force*. The magnitude of this force is $\sqrt{(\sigma\partial V/\partial x)^2 + (\sigma\partial V/\partial y)^2}$. Second order derivatives:

$$\sigma^2\frac{\partial^2 V}{\partial x^2} = +\sum_k \left[1-\left(\frac{x-x_k}{\sigma}\right)^2\right] e^{-\frac{1}{2}\left[(x-x_k)^2+(y-y_k)^2\right]/\sigma^2}$$

$$\sigma^2\frac{\partial^2 V}{\partial y^2} = +\sum_k \left[1-\left(\frac{y-y_k}{\sigma}\right)^2\right] e^{-\frac{1}{2}\left[(x-x_k)^2+(y-y_k)^2\right]/\sigma^2}$$

$$\sigma^2\frac{\partial^2 V}{\partial x\partial y} = -\sum_k \frac{(x-x_k)(y-y_k)}{\sigma^2} e^{-\frac{1}{2}\left[(x-x_k)^2+(y-y_k)^2\right]/\sigma^2}$$

Abbreviation:

$$w_k = e^{-\frac{1}{2}\left[(x-x_k)^2+(y-y_k)^2\right]/\sigma^2}$$

Lemma.

$$\left(\sum_k w_k \frac{(x-x_k)(y-y_k)}{\sigma^2}\right)^2 \le \left[\sum_k w_k \left(\frac{x-x_k}{\sigma}\right)^2\right]\left[\sum_k w_k \left(\frac{y-y_k}{\sigma}\right)^2\right]$$

Proof. This is a direct consequence of Schwartz Inequality:

$$(\vec{a}\cdot\vec{b})^2 \le (\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b}) \qquad \text{where}$$

$$\vec{a} = \left(\frac{x - x_1}{\sigma}, \frac{x - x_2}{\sigma}, \ldots, \frac{x - x_k}{\sigma}, \ldots \right)$$

$$\vec{b} = \left(\frac{y - y_1}{\sigma}, \frac{y - y_2}{\sigma}, \ldots, \frac{y - y_k}{\sigma}, \ldots \right)$$

Abbreviations:

$$N = \sum_k w_k$$

$$\mu_{xx} = \sum_k w_k \left(\frac{x - x_k}{\sigma} \right)^2 / N$$

$$\mu_{yy} = \sum_k w_k \left(\frac{y - y_k}{\sigma} \right)^2 / N$$

$$\mu_{xy} = \sum_k w_k \frac{(x - x_k)(y - y_k)}{\sigma^2} / N$$

Herewith it says:

$$\mu_{xy}^2 \le \mu_{xx}\, \mu_{yy} \quad \text{or} \quad \mu_{xx}\mu_{yy} - \mu_{xy}^2 \ge 0$$

$$\sigma^2 \frac{\partial^2 V}{\partial x^2} = N(1 - \mu_{xx}) \quad \text{and} \quad \sigma^2 \frac{\partial^2 V}{\partial y^2} = N(1 - \mu_{yy})$$

$$\sigma^2 \frac{\partial^2 V}{\partial x \partial y} = -N\mu_{xy}$$

The Jacobian matrix at (x, y) is the two-dimensional equialent of the second order derivative at that place:

$$\sigma^2 \begin{bmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial y} \\ \frac{\partial^2 V}{\partial x \partial y} & \frac{\partial^2 V}{\partial x^2} \end{bmatrix} = N \begin{bmatrix} 1 - \mu_{xx} & -\mu_{xy} \\ -\mu_{xy} & 1 - \mu_{yy} \end{bmatrix}$$

The determinant of the Jacobian matrix (i.e. the Jacobian determinant or the Jacobian) is:

$$J = N^2 \left[(1 - \mu_{xx})(1 - \mu_{yy}) - \mu_{xy}^2 \right] \quad \Longrightarrow$$

$$J = N^2 \left[(\mu_{xx}\mu_{yy} - \mu_{xy}^2) - (\mu_{xx} + \mu_{yy}) + 1 \right]$$

Where it has been established that $(\mu_{xx}\mu_{yy} - \mu_{xy}^2) \ge 0$. An ellipse of inertia can be determined at places where $J > 0$. And the major axis of that ellipse is the main direction of the pixels there. The characteristic equation for the eigenvalues λ is:

$$\left(\sigma^2 \frac{\partial^2 V}{\partial x^2} - \lambda \right) \left(\sigma^2 \frac{\partial^2 V}{\partial y^2} - \lambda \right) - \left(\sigma^2 \frac{\partial^2 V}{\partial x \partial y} \right)^2 = 0$$

Giving as a solution:

$$\lambda = \frac{\text{Sp}}{2} \pm \sqrt{\left(\frac{\text{Sp}}{2}\right)^2 - \text{Det}}$$

Where

$$\text{Sp} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \quad \text{and} \quad \text{Det} = \frac{\partial^2 V}{\partial x^2}\frac{\partial^2 V}{\partial y^2} - \left(\frac{\partial^2 V}{\partial x \partial y}\right)^2$$

The eigenvectors are found by solving one of the two (dependent) equations, for example the rst one:

$$\left(\frac{\partial^2 V}{\partial x^2} - \lambda\right) x + \frac{\partial^2 V}{\partial x \partial y}\, y = 0$$

A solution is:

$$(x, y) = \left(-\frac{\partial^2 V}{\partial x \partial y}, \frac{\partial^2 V}{\partial x^2} - \lambda\right) \quad \Longrightarrow \quad x = -\frac{\partial^2 V}{\partial x \partial y}$$

$$y = \frac{\partial^2 V/\partial x^2 - \partial^2 V/\partial y^2}{2} \pm \sqrt{\left[\frac{\partial^2 V/\partial x^2 - \partial^2 V/\partial y^2}{2}\right]^2 + \left(\frac{\partial^2 V}{\partial x \partial y}\right)^2}$$

Which reduces to:

$$(x, y) = \left(-\,\mu_{xy}\,, \frac{\mu_{xx} - \mu_{yy}}{2} \pm \sqrt{\left\{\frac{\mu_{xx} - \mu_{yy}}{2}\right\}^2 + \mu_{xy}^2}\right)$$

The solution with the plus sign (+) in $\pm$ is the normal of the direction which belongs to the greatest eigenvalue. It seems that we are nished herewith, but this is not the case. A problem is that numerical conditioning can become very bad with calculating these eigenvectors. A remedy is to use the other of the two eigenvalue equations in such cases, i.e.:

$$\frac{\partial^2 V}{\partial x \partial y}\, x + \left(\frac{\partial^2 V}{\partial y^2} - \lambda\right) y = 0$$

Which reduces to:

$$(x, y) = \left(-\frac{\mu_{xx} - \mu_{yy}}{2} \pm \sqrt{\left\{\frac{\mu_{xx} - \mu_{yy}}{2}\right\}^2 + \mu_{xy}^2}, \, -\,\mu_{xy}\right)$$

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch.