Fuzzy Analysis

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Convolution integral

Let f be an integrable function. Consider the *fuzzyfied* function or *fuzzyfication* noted as $\overline{f}$, which is defined by the following convolution integral:

$$\overline{f}(x) := \int_{-\infty}^{+\infty} f(\xi)\,\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\xi)^2/\sigma^2}\,d\xi$$

Theorem.

$$\overline{f}(x) \approx f(x) + \frac{1}{2}\sigma^2 f''(x)$$

Proof. Substitute $\xi - x = u$ in:

$$\int_{-\infty}^{+\infty} f(\xi)\,\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\xi)^2/\sigma^2}\,d\xi = \int_{-\infty}^{+\infty} f(u+x)\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}u^2/\sigma^2}\,du =$$

$$\int_{-\infty}^{+\infty}\left[f(x) + f'(x).u + \frac{1}{2}.f''(x).u^2 + h.o.t.\right]\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}u^2/\sigma^2}\,du \approx$$

$$f(x)\int_{-\infty}^{+\infty}\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}u^2/\sigma^2}\,du \;+\; f'(x)\int_{-\infty}^{+\infty} u\,\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}u^2/\sigma^2}\,du$$

$$+\frac{1}{2}f''(x)\int_{-\infty}^{+\infty} u^2\,\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}u^2/\sigma^2}\,du = f(x).1 + f'(x).0 + \frac{1}{2}f''(x).\sigma^2$$

Corrolary. The fuzzyfication $\overline{f}$ is greater than the function f itself where the function is convex. The fuzzyfication is smaller than the function itself where the function is concave.
Theorem.

$$\int_{-\infty}^{+\infty}\overline{f}(x)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx$$

Proof.

$$\int_{-\infty}^{+\infty}\overline{f}(x)\,dx = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\xi)\,\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\xi)^2/\sigma^2}\,d\xi\right]dx =$$

$$\int_{-\infty}^{+\infty} f(\xi)\left[\int_{-\infty}^{+\infty}\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\xi)^2/\sigma^2}\,dx\right]d\xi = \int_{-\infty}^{+\infty} f(\xi).1\,d\xi$$

Corrolary. This means that, nevertheless, the overall behaviour of the fuzzyfication $\overline{f}$ is such that being greater or smaller than f is only a local phenomenon and cancels out globally.
Theorem. For integrable functions f which are *bounded* :

$$\frac{d}{dx}\overline{f}(x) = \overline{\left(\frac{df}{dx}\right)}$$

Then the derivative of the fuzzyfication is the fuzzyfication of the derivative.
Proof. The other way around:

$$\int_{-\infty}^{+\infty} f'(\xi)\,\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\xi)^2/\sigma^2}\,d\xi = \int_{-\infty}^{+\infty}\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\xi)^2/\sigma^2}\,df(\xi) =$$

$$\left[\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\xi)^2/\sigma^2}f(\xi)\right]_{\xi=-\infty}^{\xi=+\infty} - \int_{-\infty}^{+\infty} f(\xi)\,\frac{d}{d\xi}\,\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\xi)^2/\sigma^2}\,d\xi =$$

$$0\ -\ \int_{-\infty}^{+\infty} f(\xi)\left[-\frac{d}{dx}\right]\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\xi)^2/\sigma^2}\,d\xi =$$

$$\frac{d}{dx}\int_{-\infty}^{+\infty} f(\xi)\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\xi)^2/\sigma^2}\,d\xi$$

Corrolary. The derivative of the fuzzyfication is:

$$\int_{-\infty}^{+\infty} f(\xi)\frac{d}{dx}\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\xi)^2/\sigma^2}\,d\xi =$$

$$\int_{-\infty}^{+\infty} f(\xi)\left[-\frac{x-\xi}{\sigma^2}\right]\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\xi)^2/\sigma^2}\,d\xi$$

and can be calculated *independent* of the derivative $f'(x)$. Even better. The fuzzyfication of a function that is *not* differentiable at all nevertheless shall be differentiable.

Sampling theorem

It will become clear, in the end, that the theory below may be considered as a Fuzzy version of the famous Sampling Theorem by Claude Shannon.
Recall the definition of the fuzzyfication $\overline{f}$ of a function f:

$$\overline{f}(x) := \int_{-\infty}^{+\infty} f(\xi)\,\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\xi)^2/\sigma^2}\,d\xi$$

Definition. The (continuous) Fourier transform of the fuzzyfication is:

$$\overline{F}(\omega) = \int_{-\infty}^{+\infty}\overline{f}(x)\,e^{-i\,\omega x}\,dx$$

And it will be studied now:

$$\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(\xi) \, \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{1}{2}(x-\xi)^2/\sigma^2} \, d\xi \right] e^{-i\,\omega x} \, dx =$$

$$\int_{-\infty}^{+\infty} f(\xi) \, \frac{1}{\sigma\sqrt{2\pi}} \, \left[\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x-\xi)^2/\sigma^2 - i\,\omega x} \, dx \right] d\xi$$

The exponent of the integral between square brackets [ ] is worked out:

$$-\frac{1}{2}\left(\frac{x-\xi}{\sigma}\right)^2 - i\,\omega x =$$

$$-\frac{1}{2}\left[\left(\frac{x-\xi}{\sigma}\right)^2 + 2\,i\,\omega\,\sigma \left(\frac{x-\xi}{\sigma}\right) + (i\,\omega\,\sigma)^2 \right] - i\,\omega\,\xi + \frac{1}{2}(i\,\omega\,\sigma)^2 =$$

$$-\frac{1}{2}\left[\left(\frac{x-\xi}{\sigma}\right) + i\,\omega\,\sigma \right]^2 - i\,\omega\,\xi - \frac{1}{2}\omega^2\sigma^2$$

Hence:

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x-\xi)^2/\sigma^2 - i\,\omega x} \, dx = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[\left(\frac{x-\xi}{\sigma}\right) + i\,\omega\,\sigma\right]^2} \, dx \; . \; e^{-i\,\omega\,\xi} . e^{-\frac{1}{2}\omega^2\sigma^2}$$

$$= \sigma\,\sqrt{2\pi} \; . \; e^{-i\,\omega\,\xi} . e^{-\frac{1}{2}\omega^2\sigma^2}$$

Theorem.

$$\overline{F}(\omega) = e^{-\frac{1}{2}\omega^2\sigma^2} \int_{-\infty}^{+\infty} f(\xi) \, e^{-i\,\omega\,\xi} \, d\xi \; = e^{-\frac{1}{2}\omega^2\sigma^2} F(\omega)$$

Proof. According to the above. Not quite unexpected, because the Fourier transform of a convolution of the functions g and f is equal to the Fourier transform of g times the Fourier transform of f. And we know that the Fourier transform of the former is given by $\exp(-\frac{1}{2}\,\omega^2\sigma^2)$.

It is assumed that the Fourier transform of f is bounded: $|F(\omega)| < M$. Gauss functions are rapidly decreasing for increasing values of their independent variable. Thus ω can be chosen in such a way ($= \omega_g$) that the absolute value of $\exp(-\frac{1}{2}\,\omega^2\sigma^2)F(\omega)$ will become neglectable:

$$e^{-\frac{1}{2}\sigma^2\omega_g^2} M < \epsilon \quad \Longleftrightarrow \quad -\frac{1}{2}\sigma^2\omega_g^2 < ln(\epsilon/M) \quad \Longleftrightarrow$$

$$\sigma^2\omega_g^2 > 2.ln(M/\epsilon) \quad \Longleftrightarrow \quad \omega_g > \sqrt{2.ln(M/\epsilon)}\;/\;\sigma$$

Suppose now that we choose to make a periodic function out of $\overline{F}(\omega)$, simply by repeating this "pulse" with a period greater than $2.\omega_g$. This new function can be developed into a Fourier Series:

$$\overline{F}_T(\omega) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k\, cos(k\frac{2\pi}{T}\omega) + B_k\, sin(k\frac{2\pi}{T}\omega)$$

Where:

$$\frac{1}{2}(A_k - iB_k) = \frac{1}{T}\int_{-\frac{1}{2}T}^{+\frac{1}{2}T} \overline{F}(\omega)e^{+i\,(k2\pi/T)\,\omega}\, d\omega = \frac{1}{T}\int_{-\infty}^{+\infty} \overline{F}(\omega)e^{i\,(k2\pi/T)\,\omega}\, d\omega$$

Replacement of $[-T/2, +T/2]$ by $[-\infty, +\infty]$ is possible because it is assumed that the period T is greater than $2\,\omega_g$. Therefore the extension of the interval to infinity should provide a contribution which is neglectable. Now remember the original definition of the (continuous) Fourier transform:

$$\overline{F}(\omega) = \int_{-\infty}^{+\infty} \overline{f}(x)\, e^{-i\,\omega x}\, dx$$

It must have an inverse, which is found to be:

$$\overline{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{F}(\omega)\, e^{i\,\omega x}\, d\omega$$

Thus it is seen that:

$$\frac{1}{2}(A_k + iB_k)^* = \frac{2\pi}{T}\,\overline{f}\left(k\frac{2\pi}{T}\right)$$

We find that the complex conjugate of the (complex) Fourier coefficients (apart from a constant) is equal to a sampling of the fuzzyfication. For the sampling frequency $2\pi/T$ we find:

$$\Delta = \frac{2\pi}{T} < \frac{\pi}{\omega_g}$$

But:

$$\omega_g > \sqrt{2.ln(M/\epsilon)}\,/\,\sigma$$

Hence:

$$\Delta < \frac{\pi}{\sqrt{2.ln(M/\epsilon)}}\,\sigma$$

Thus, with such a sampling Δ, which has to be considered as a discretization of our continuous fuzzyfication, the Fourier coefficients of a series $\overline{F}_T(\omega)$ can be constructed. Then we extract from the accompanying periodic function only one period, the function $\overline{F}(\omega)$, by simply deleting all periods except one. At last, this "central pulse" can be transformed back to the continuous fuzzyfication itself. We thus find that the fuzzyfication is entirely determined by its sampling, provided that such a sampling is at intervals smaller than the abovementioned value, which in turn is related to the spread σ by an amount:

$$\Delta < \frac{\pi}{\sqrt{2.ln(M/\epsilon)}}\,\sigma$$

Here ϵ/M is the desired accuracy for exponential (Gauss) functions. It is noticed that the sampling width Δ is varying very *slowly* with the latter quantity while it is varying linearly with the spread σ. Furthermore:

$$\epsilon/M = e^{-2.\pi^2} \quad \Longleftrightarrow \quad \frac{\pi}{\sqrt{2.ln(M/\epsilon)}} = \frac{1}{2} \quad \Longleftrightarrow \quad \Delta < \frac{1}{2}\sigma$$

$$\text{where} \quad e^{-2.\pi^2} \approx 2.67528799107424\,.\,10^{-9}$$

So, using a sampling distance of less than or equal to *half* the spread gives rise to a truncation error which is less than three per billion. Let's summarize our findings in a **Theorem**. Well, later ...

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch.