Trace Weighting

It is known from litterature that the Least Squares Finite Element "variational integral" can be modified a little bit as follows, according to Zienkiewicz [1] chapter 3.14.2 equation (3.168):

$$\iint \left\{ A.\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]^2 + B.\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]^2 \right\} dx.dy = \text{minimum}$$

Here A and B are "positive valued functions or constants", which may be chosen in a convenient way. Quoting without permission from Zienkiewicz: *Once again this weighting function could be chosen as to ensure a constant ratio of terms contributed by various elements - although this has not yet been put into practice.* Well, somebody has to be the first ...

Suppose that we plan to adhere to this practice indeed. Then it should be remarked immediately that, for example, the area of an element is not relevant anymore: it is "absorbed" into the constants A and B, which are arbitrary. So constants like J and w may be left out altogether; appropriate weighting factors may be chosen differently instead. Let's jump to the discretized equivalent of the above:

$$\sum_{k=1}^{4} \left(\sum_{j=1}^{8} A_{k,j}\, w_j \right)^2 = \text{minimum}(w_j) = 0$$

Ziekiewicz' weighting procedure translates into the notion that coefficients of the finite difference matrix rows A_k are determined up to an arbitrary constant. This means that it is admissible to multiply every row with a certain (positive) number α_k, for the purpose of optimizing something:

$$\sum_{k=1}^{4} \left(\sum_{j=1}^{8} \alpha_k \, A_{k,j} \, w_j \right)^2 = \text{minimum}(w_j) = 0$$

This additional degree of freedom is a good thing, because herewith a remedy may be found for still another tricky phenomenon, associated with L.S.FEM: taking the square of a system of equations will also square the *condition number* of that system. I'm not going to explain why the condition number of a matrix is the quotient of its greatest and its smallest eigenvalue. Some decent references can be found on the Web:

`http://www.math.gatech.edu/~bourbaki/math2601/Web-notes/1num.pdf`

Now, if the eigenvalues of the original matrix are given by λ, then the eigenvalues of the squared matrix (that is: the transpose multiplied by the original) are given by the squares of the absolute values of the same λ:

$$A\,x = \lambda\,x \quad \Longrightarrow \quad A^T A\,x = \lambda^* \lambda\,x = |\lambda|^2\,x$$

The condition number $C(A)$ of a matrix A is defined as the absolute value of its largest eigenvalue $\lambda_{max}(A)$ divided by its smallest eigenvalue $\lambda_{min}(A)$. It can be demonstrated that the condition number is kind of a measure for the loss of decimals with operations like inverting the matrix. Therefore a large condition number is considered to be bad. It is easily shown that the condition number of a squared system of equations is quadratically *worse*, when compared with the condition number of the original system:

$$C(A) = \frac{\lambda_{max}}{\lambda_{min}} \quad \Longrightarrow \quad C(A^T A) = \left| \frac{\lambda_{max}}{\lambda_{min}} \right|^2$$

Hence it is clear that attention should be given to the condition of equations emerging from a least squares procedure. Putting the idea of Zienkiewicz into practice, a proper choice for the weighting factors α_k indeed should be employed for optimizing the condition of the global Least Squares matrix. It's common practice to arrange things in such a way that the contributions of all rows become approximately the same. Probably the easiest way to accomplish this would be: to divide every F.D. Equation (k) by its own "length". Meaning that $\alpha_k = 1/L_k$ in:

$$\sum_{j=1}^{8} \frac{A_{k,j}}{L_k} \, w_j \qquad \text{with} \quad L_k = \sqrt{\sum_i A_{k,i}^2} \qquad k = 1, \ldots, 4$$

Now take a look at the element matrix E which is associated with *one* of the equations (k) in the associated finite difference problem:

$$E_{i,j}^{(k)} = A_{k,i}\,A_{k,j} \quad \Longrightarrow \quad E_{i,i}^{(k)} = A_{k,i}^2 \quad \Longrightarrow \quad Sp\left[E^{(k)}\right] = \sum_i A_{k,i}^2 = L_k^2$$

Therefore the coefficients $\sum_i A_{k,i}^2$ are also found as the *trace Sp* of the element matrix belonging to one of the finite difference equations. Thus the weighting procedure can also be accomplished by dividing all coefficients of a "single" element matrix by the *trace* of this matrix. Hence the name: *trace weighting* or "SpoorWegen" (Pun: "RailRoads" in Dutch. And don't confuse weight tracing as opposed to ray tracing ... ;-)
How about the following little **theorem**: *The trace of the whole system's matrix is equal to the total number of finite difference equations involved.* Providing the programmer with a means to check out whether the requirement of balancing the (normed) F.D. equations is actually fulfilled.

References

[1] O.C. Zienkiewicz,
'The Finite Element Method', 3th edition, Mc.Graw-Hill U.K. 1977, ISBN 0-07-084072-5