About Inside/Outside Problems

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Sharpened Points and Contours

A point in the plane is an idealized pixel, without size. A pixel is a fuzzyfied point, smeared out over a small domain in the plane. Cast in more mathematical terms: a point at (p, q) is a delta function at that place. The fuzzyfication of that point, the pixel, can be modelled by a 2-D bell shaped function, which is centered at (p, q):

$$\begin{aligned} \text{point} &= \delta(x-p, y-q) \quad = \quad \delta(x-p)\delta(y-q) \\ \text{fuzzy} &= \frac{1}{2\pi\sigma^2}\, e^{-\frac{1}{2}\left[(x-p)^2+(y-q)^2\right]/\sigma^2} \\ &= \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(x-p)^2/\sigma^2} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(y-q)^2/\sigma^2} \end{aligned}$$

The point can be fuzzyfied to a pixel and the pixel can be sharpened to a point. The latter is accomplished by a limit where the spread called σ becomes zero:

$$\lim_{\sigma\to 0} \frac{1}{2\pi\sigma^2}\, e^{-\frac{1}{2}\left[(x-p)^2+(y-q)^2\right]/\sigma^2} =$$

$$\lim_{\sigma\to 0} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(x-p)^2/\sigma^2} \lim_{\sigma\to 0} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(y-q)^2/\sigma^2}$$

$$= \delta(x-p)\delta(y-q) = \delta(x-p, y-q)$$

The above becomes somewhat less trivial as soon as (the fuzzyfication of) a point is integrated over a certain area, which is enclosed by a contour curve:

$$\iint \delta(x-p, y-q)\, dx\, dy$$

The outcome of this integral depends on whether the point (p, q) is *inside* or *outside* the contour curve which encloses the area. In the former case it is obviously one, in the latter case it is zero:

$$\iint \delta(x-p, y-q)\, dx\, dy = \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right. \text{ if } (p,q) \begin{array}{c} \text{inside} \\ \text{outside} \end{array}$$

If the delta function is "cut into pieces", which happens if (p, q) is exactly at the boundary curve, then the outcome of the integral is a *undefined* value, though somewhere between 0 and 1 . But let's discard this special case for the moment being. And consider instead the fuzzyfication of the above:

$$\iint \frac{1}{2\pi\sigma^2}\, e^{-\frac{1}{2}\left[(x-p)^2+(y-q)^2\right]/\sigma^2}\, dx\, dy$$

$$= \iint \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}(x-p)^2/\sigma^2}\, \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}(y-q)^2/\sigma^2}\; dx\, dy$$

According to Green's Theorem, any such area integral may be converted into a contour integral. As follows:

$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\, dx\, dy = \oint (P\, dx + Q\, dy)$$

Where the line integral must be evaluated along the contour enclosing the area of interest. Converting a line integral into an area integral is easy. But the reverse, converting an area integral into a line integral, requires some more ingenuity. In our case, though, there exists more than one possibility to do it. We opt for the following. Let the function $P(x,y)$ be zero everywhere and let the function $Q(x,y)$ be defined by:

$$Q(x,y) = \mathit{Erf}\left(\frac{x-p}{\sigma}\right)\, \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}(y-q)^2/\sigma^2} \qquad \Longrightarrow$$

$$\frac{\partial Q}{\partial x} = \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}(x-p)^2/\sigma^2}\, \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}(y-q)^2/\sigma^2}$$

Which is exactly what we want. Now the result of Green's Theorem becomes:

$$\iint \frac{\partial Q}{\partial x}\, dx\, dy = \oint Q\, dy = \oint \mathit{Erf}\left(\frac{x-p}{\sigma}\right)\, \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}(y-q)^2/\sigma^2}\; dy$$

In the limiting case, when $\sigma \to 0$, the exponential function becomes a delta function and the *Erf* function becomes an integrated delta function, namely the Heaviside function. Thus we find:

$$\lim_{\sigma\to 0} \oint \mathit{Erf}\left(\frac{x-p}{\sigma}\right)\, \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}(y-q)^2/\sigma^2}\; dy = \oint H(x-p)\, \delta(y-q)\, dy$$

But there are a myriad other ways to look at the integral:

$$\iint \frac{1}{2\pi\sigma^2}\; e^{-\frac{1}{2}\left[(x-p)^2+(y-q)^2\right]/\sigma^2}\; dx\, dy$$

For example, take the origin of your coordinates at $(p,q) = (0,0)$ and replace the Cartesian (x,y) system by polar coordinates (r,ϕ) :

$$\iint \frac{1}{2\pi\sigma^2}\; e^{-\frac{1}{2}\left[x^2+y^2\right]/\sigma^2}\; dx\, dy = \iint \frac{1}{2\pi\sigma^2}\; e^{-\frac{1}{2}r^2/\sigma^2}\; r\, dr\, d\phi$$

The latter integral can be evaluated as follows:

$$\iint \frac{1}{2\pi\sigma^2}\; e^{-\frac{1}{2}r^2/\sigma^2}\; r\, dr\, d\phi = \frac{1}{2\pi} \int_\phi \left[- \int_r e^{-\frac{1}{2}(r/\sigma)^2}\, d\left\{ -\frac{1}{2}(r/\sigma)^2 \right\} \right] d\phi$$

If we take the limit for $\sigma \to 0$ of the integral between square brackets, then it is reduced to unity:

$$\lim_{\sigma \to 0} - \int_0^{-\frac{1}{2}R^2/\sigma^2} e^t \, dt = \lim_{\sigma \to 0} \left[1 - e^{-\frac{1}{2}(R/\sigma)^2} \right] = 1$$

Here R denotes the distance of the boundary curve to the origin. If it may be assumed that the origin is *distinct from the boundary*, then for $\sigma \to 0$ the quotient R/σ approaches infinity and the exponential function becomes zero indeed. So we are left with:

$$\iint \frac{1}{2\pi\sigma^2} \, e^{-\frac{1}{2}r^2/\sigma^2} \, r \, dr \, d\phi = \frac{1}{2\pi} \oint d\phi$$

Where the integral at the right is recognized as the *winding number* of the curve enclosing the area of interest. This leads to the following concise conclusion:

$$\boxed{\oint H(x) \, \delta(y) \, dy = \frac{1}{2\pi} \oint d\phi}$$

Where the point of interest, formerly called (p, q) , is identified with the origin $(0, 0)$ on a permanent basis. I feel that this conclusion is valid for an arbitrary bunch of mutually and eventually self-intersecting *closed* curves. With the sole condition that the origin is not coincident with any place on these.
The rest of the story is technology, not a theory. I want to spend a few words on it, though. The integral with $H(x)$ and $\delta(y)$ in it is actually equivalent with the following well-known statement. Draw a line from the point of interest P, in the positive x-direction, towards infinity. Then count the number of intersections of that line with the curves that form the boundary of the area of interest A. If the number of intersections is *odd* then P is *inside* A. If the number of intersections is *even* then P is *outside* A. However, the technical implementation of this is far more tricky that one might expect. Special cases like being smaller ($<$), equal ($=$) or greater ($>$) than zero should be distinguished carefully, eventually resulting in $3^2 \times 3^2 = 81$ different combinations (!). It should be mentioned, as well, that the Heaviside function $H(x)$ algorithmically means that the infamous *clipping problem* (see: Computer Graphics) has an instance at $x = 0$. Last but not least, if you really want to improve efficiency, then a *sorting and searching problem* with respect to the y-coordinates will become part of the project.

Fuzzyfication of Line Segments

Let the line segment l between (x_0, y_0) and (x_1, y_1) be given by:

$$x(t) = x_0 + (x_1 - x_0).t \quad \text{and} \quad y(t) = y_0 + (y_1 - y_0).t \qquad \text{where:} \quad 0 < t < 1$$

The fuzzyfied line segment L is defined by a convolution integral of l with the standard normal distribution, in two dimensions:

$$L(x, y) = \iint \left(\frac{1}{\sigma\sqrt{2\pi}} \right)^2 e^{-\frac{1}{2}\left[(\xi - x)^2 + (\eta - y)^2\right]/\sigma^2} l(\xi, \eta)\, d\xi.d\eta$$

Here $l(x, y)$ denotes the line segment. It is advantageous to introduce other coordinates, which are associated with l itself. The parameter t is one of these coordinates. Let the thickness of the line segment be denoted by D and the length measured along l with s, then $s = t.\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$ and:

$$\xi = x_0 + (x_1 - x_0).t \qquad \eta = y_0 + (y_1 - y_0).t$$

$$d\xi.d\eta = D.ds = D.\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}\, dt$$

Furthermore, the function $l(x, y)$ has a value 1 at the line segment (and 0 everywhere else). Herewith, the double integral becomes:

$$L(x, y) = \frac{1}{2\pi\sigma^2} D.\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

$$. \int_0^1 e^{-\frac{1}{2}\left\{[x_0 + (x_1 - x_0).t - x]^2 + [y_0 + (y_1 - y_0).t - y]^2\right\}/\sigma^2} \;.\; 1 \;.\; dt$$

So that is what we have to calculate. Start rewriting the exponent:

$$[x_0 + (x_1 - x_0).t - x]^2 + [(y_0 + (y_1 - y_0).t - y]^2 =$$

$$\left[(x_1 - x_0)^2 + (y_1 - y_0)^2\right].t^2 - 2.\left[(x - x_0)(x_1 - x_0) + (y - y_0)(y_1 - y_0)\right].t$$

$$+ \left[(x - x_0)^2 + (y - y_0)^2\right] = A.t^2 - 2.B.t + C$$

Where:

$$A = (x_1 - x_0)^2 + (y_1 - y_0)^2$$
$$B = (x - x_0)(x_1 - x_0) + (y - y_0)(y_1 - y_0)$$
$$C = (x - x_0)^2 + (y - y_0)^2$$

Write as follows: $A.t^2 - 2.B.t + C =$

$$A\left[t^2 - 2\frac{B}{A} + \left(\frac{B}{A}\right)^2\right] - A\left(\frac{B}{A}\right)^2 + C = A\left(t - \frac{B}{A}\right)^2 - \frac{B^2 - A.C}{A}$$

Simplify with:

$$X = x - x_0 \qquad Y = y - y_0 \qquad X_1 = x_1 - x_0 \qquad Y_1 = y_1 - y_0$$

Giving:

$$B^2 - A.C = [(x-x_0)(x_1-x_0)+(y-y_0)(y_1-y_0)]^2$$
$$-\left[(x_1-x_0)^2+(y_1-y_0)^2\right]\left[(x-x_0)^2+(y-y_0)^2\right]$$
$$= (X.X_1+Y.Y_1)^2 - (X_1+Y_1)^2(X^2+Y^2)$$
$$= X^2.X_1^2 + 2.X.X_1.Y.Y_1 + Y^2.Y_1^2 - X_1^2.X^2 - X_1^2.Y^2 - Y_1^2.X^2 - Y_1^2.Y^2$$
$$= -X_1^2.Y^2 + 2.X_1.Y.Y_1.X - Y_1^2.X^2 = -(X_1.Y - Y_1.X)^2$$
$$\Longrightarrow \quad -\frac{B^2 - A.C}{A} = \frac{[(x_1-x_0)(y-y_0)-(y_1-y_0)(x-x_0)]^2}{(x_1-x_0)^2+(y_1-y_0)^2}$$

Herewith the exponent becomes: $A.t^2 - 2.B.t + C =$

$$\left[(x_1-x_0)^2+(y_1-y_0)^2\right]\left[t - \frac{(x-x_0)(x_1-x_0)+(y-y_0)(y_1-y_0)}{(x_1-x_0)^2+(y_1-y_0)^2}\right]^2$$
$$+\frac{[(x_1-x_0)(y-y_0)-(y_1-y_0)(x-x_0)]^2}{(x_1-x_0)^2+(y_1-y_0)^2}$$

Define position vectors:

$$\vec{r} = (x,y) \qquad \vec{r}_0 = (x_0,y_0) \qquad \vec{r}_1 = (x_1,y_1)$$

Inner products ($\cdot$) and the absolute value of an outer product ($\times$) may be easily recognized:

$$(x-x_0)(x_1-x_0)+(y-y_0)(y_1-y_0) = (\vec{r}_1 - \vec{r}_0 \cdot \vec{r} - \vec{r}_0)$$
$$(x_1-x_0)^2+(y_1-y_0)^2 = (\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)$$
$$|(x_1-x_0)(y-y_0)-(y_1-y_0)(x-x_0)| = |(\vec{r}_1-\vec{r}_0)\times(\vec{r}-\vec{r}_0)|$$

Herewith: $A.t^2 - 2.B.t + C =$

$$(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)\left[t - \frac{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r} - \vec{r}_0)}{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}\right]^2 + \frac{|(\vec{r}_1-\vec{r}_0)\times(\vec{r}-\vec{r}_0)|^2}{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}$$

Rewrite:

$$L(x,y) = \frac{1}{2\pi\sigma^2}\, D.\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}\ \int_0^1 e^{-\frac{1}{2}\left\{A.t^2-2.B.t+C\right\}/\sigma^2}\, dt$$

$$= \frac{1}{2\pi\sigma^2}\, D.\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}\ .\ e^{-\frac{1}{2}\left\{|(\vec{r}_1-\vec{r}_0)\times(\vec{r}-\vec{r}_0)|^2/(\vec{r}_1-\vec{r}_0\cdot\vec{r}_1-\vec{r}_0)\right\}/\sigma^2}$$
$$\int_0^1 e^{-\frac{1}{2}\left\{(\vec{r}_1-\vec{r}_0\cdot\vec{r}_1-\vec{r}_0)[t-(\vec{r}_1-\vec{r}_0\cdot\vec{r}-\vec{r}_0)/(\vec{r}_1-\vec{r}_0\cdot\vec{r}_1-\vec{r}_0)]^2\right\}/\sigma^2}\, dt$$

Thus, the exponential function splits up into a part which is quite independent of the running parameter t and another part which is still dependent on it. Only the latter has to be integrated further, of course. For that purpose, introduce a new variable u:

$$u = \left(\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)} \, . \, t - \frac{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r} - \vec{r}_0)}{\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} \right) / \sigma$$

$$\Longrightarrow \quad du = \sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)} \, . \, dt/\sigma \quad \Longrightarrow \quad dt = \frac{\sigma}{\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} du$$

And the integral becomes:

$$\frac{\sigma\sqrt{2\pi}}{\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} \frac{1}{\sqrt{2\pi}} \int_{u_0}^{u_1} e^{-\frac{1}{2}u^2} du$$

Where:

$$u_0 = -\frac{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r} - \vec{r}_0)}{\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} / \sigma$$

And:

$$u_1 = \left(\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)} - \frac{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r} - \vec{r}_0)}{\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} \right) / \sigma$$

$$= \frac{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0) - (\vec{r}_1 - \vec{r}_0 \cdot \vec{r} - \vec{r}_0)}{\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} / \sigma = \frac{(\vec{r}_1 - \vec{r} \cdot \vec{r}_1 - \vec{r}_0)}{\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} / \sigma$$

Summarizing:

$$u_0 = -\frac{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r} - \vec{r}_0)}{\sigma\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} \quad \text{and} \quad u_1 = -\frac{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r} - \vec{r}_1)}{\sigma\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}}$$

The integral over a normal distribution can always be expressed as the sum of two error functions, where the ERror Function (Erf) is defined as:

$$Erf(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} \, dt \quad \Longrightarrow \quad \frac{1}{\sqrt{2\pi}} \int_{u_0}^{u_1} e^{-\frac{1}{2}u^2} \, du = Erf(u_1) - Erf(u_0)$$

Thus, with the values involved, the integral to be calculated turns out to be equal to:

$$\frac{\sigma\sqrt{2\pi}}{\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} \left[Erf(u_1) - Erf(u_0) \right] =$$

$$\frac{\sigma\sqrt{2\pi}}{\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} \left[Erf(-u_0) - Erf(-u_1) \right] = \frac{\sigma\sqrt{2\pi}}{\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}}$$

$$. \left[Erf\left(\frac{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r} - \vec{r}_0)}{\sigma\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} \right) - Erf\left(\frac{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r} - \vec{r}_1)}{\sigma\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} \right) \right]$$

There are two factors in front of the end result, which partly cancel out:

$$\frac{1}{2\pi\sigma^2}\ D.\sqrt{(\vec{r}_1-\vec{r}_0\cdot\vec{r}_1-\vec{r}_0)}\frac{\sigma\sqrt{2\pi}}{\sqrt{(\vec{r}_1-\vec{r}_0\cdot\vec{r}_1-\vec{r}_0)}}=\frac{D}{\sigma\sqrt{2\pi}}$$

Therefore the final result must read:

$$L(x,y)=\frac{D}{\sigma\sqrt{2\pi}}\ .\ e^{-\frac{1}{2}\left\{|(\vec{r}_1-\vec{r}_0)\times(\vec{r}-\vec{r}_0)|^2/(\vec{r}_1-\vec{r}_0\cdot\vec{r}_1-\vec{r}_0)\right\}/\sigma^2}$$

$$.\ \left[Erf\left(\frac{(\vec{r}_1-\vec{r}_0\cdot\vec{r}-\vec{r}_0)}{\sigma\sqrt{(\vec{r}_1-\vec{r}_0\cdot\vec{r}_1-\vec{r}_0)}}\right)-Erf\left(\frac{(\vec{r}_1-\vec{r}_0\cdot\vec{r}-\vec{r}_1)}{\sigma\sqrt{(\vec{r}_1-\vec{r}_0\cdot\vec{r}_1-\vec{r}_0)}}\right)\right]$$

Having arrived at the end of the story, let's put everything the other way around. Instead of fuzzying a line segment, leave it as it is. Consider instead the influence of a fuzzy point in the plane upon an exact line segment, that is: integrate the bell shaped function $exp(-\frac{1}{2}(\xi^2+\eta^2)/\sigma^2)$ of the fuzzy pixel over all points of the line segment. Here ξ and η are the components of the vector joining the pixel with any point of the line segment. Setting up the mathematics results in:

$$\frac{1}{\sigma\sqrt{2\pi}}\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}$$

$$.\ \int_0^1 e^{-\frac{1}{2}\left\{[x_0+(x_1-x_0).t-x]^2+[y_0+(y_1-y_0).t-y]^2\right\}/\sigma^2}\ dt$$

Apart from a constant factor, this is completely equivalent with working the other way around. So it makes hardly any difference, if at all, whether a fuzzy line segment is sensed by an exact pixel or whether a fuzzy pixel is sensed by an exact line segment. Thus it makes no difference whether the *theory or the experiment* is fuzzyfied, as long as *one of both is the fuzzy one*.

Fuzzyfied Lines Unclipped

The final formula from the previous paragraph is repeated here for convenience:

$$L(x,y)=\frac{D}{\sigma\sqrt{2\pi}}\ .\ e^{-\frac{1}{2}\left\{|(\vec{r}_1-\vec{r}_0)\times(\vec{r}-\vec{r}_0)|^2/(\vec{r}_1-\vec{r}_0\cdot\vec{r}_1-\vec{r}_0)\right\}/\sigma^2}$$

$$.\ \left[Erf\left(\frac{(\vec{r}_1-\vec{r}_0\cdot\vec{r}-\vec{r}_0)}{\sigma\sqrt{(\vec{r}_1-\vec{r}_0\cdot\vec{r}_1-\vec{r}_0)}}\right)-Erf\left(\frac{(\vec{r}_1-\vec{r}_0\cdot\vec{r}-\vec{r}_1)}{\sigma\sqrt{(\vec{r}_1-\vec{r}_0\cdot\vec{r}_1-\vec{r}_0)}}\right)\right]$$

There exists a sensible interpretation of the factor with the error functions:

$$\frac{(\vec{r}_1-\vec{r}_0\cdot\vec{r}-\vec{r}_{0,1})}{\sqrt{(\vec{r}_1-\vec{r}_0\cdot\vec{r}_1-\vec{r}_0)}}$$

According to the theory of inner products, these terms denote a *projection*, along the line segment $(\vec{r}_1 - \vec{r}_0)$, of the vectors joining the point $\vec{r}$ in the plane, with the startpoint $\vec{r}_0$ or endpoint $\vec{r}_1$, respectively. This could also be expressed as follows:

$$s = \frac{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r} - \vec{r}_0)}{\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} \quad \Longrightarrow$$

$$\frac{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r} - \vec{r}_1)}{\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} = \frac{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r} - \vec{r}_0)}{\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} - \frac{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}{\sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}} = s - S$$

Where $S = \sqrt{(\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)}$ is the total length of the segment. Thinking in this way results in a term:

$$[Erf(s) - Erf(s - S)]$$

The difference of the two error functions assumes the value 1 for most of the line segment L. The $Erf()$ part assumes appreciably different values only in the neighbourhood of the start- and endpoints. That is, if the projection of the point in the plane upon the segment l gives rise to a distance, measured along the segment, of less than a few times the spread, from the end-points $(\vec{r}_0, \vec{r}_1)$. This is one reason why the term with the $Erf()$ functions is relatively less important and can be set to 1 in many cases. To put it in different terms, instead of integrating the line segment over its finite length, it could be decided to extend the integration interval to infinity. This corresponds with $(s, s-S) = (+\infty, -\infty)$. For these values of s, it is known that $Erf(s) = 1$ and $Erf(s-S) = 0$. With infinite length, only the length independent part of the problem is left. However, in order to get rid of the $Erf()$ terms, there is no need to go all the way to infinity. Instead, it is sufficient to restrict the area of interest to a domain which is sufficiently far away from the end-points of the straight line segments. Here, *sufficiently far away* may be conveniently defined as a couple of times the value of the spread. A nice (and easy to remember) value is with 2π :

$$2\pi\sigma \quad \Longrightarrow \quad e^{-\frac{1}{2}(2\pi\sigma)^2/\sigma^2} = e^{-2\pi^2} (\approx 2.67528799107424 \,.\, 10^{-9})$$

One way to accomplish things is to augment the area of interest with a kind of sufficiently wide *margin*, where the width of the margin may be equal to, for example, the above $2\pi\sigma$. Anyway, it's not very difficult to eliminate, in practice, the factor with the $Erf()$ terms in it. And we find the fuzzyfication of a straight line, unfinished, unclipped:

$$L(x, y) = \frac{D}{\sigma\sqrt{2\pi}} \,.\, e^{-\frac{1}{2}\left\{|(\vec{r}_1 - \vec{r}_0) \times (\vec{r} - \vec{r}_0)|^2 / (\vec{r}_1 - \vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0)\right\}/\sigma^2}$$

Or:

$$L(x, y) = \frac{D}{\sigma\sqrt{2\pi}} \,.\, e^{-\frac{1}{2}(B/\sigma)^2}$$

Where:

$$B = \frac{|(x-x_0)(y_1-y_0)-(x_1-x_0)(y-y_0)|}{\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}}$$

A lucid interpretation for the latter quantity does also exist. The nominator is equal to the absolute value of the outer product $(\vec{r}_1 - \vec{r}_0) \times (\vec{r} - \vec{r}_0)$. This, in turn, is equal to the area of the paralellogram which is spanned by the vectors $(\vec{r}_1 - \vec{r}_0)$ and $(\vec{r} - \vec{r}_0)$. The paralellogram can be divided in two congruent triangles: $(\vec{r}_0, \vec{r}_1, \vec{r})$ and $(\vec{r}, \vec{r}_1, \vec{r}_1 + \vec{r} - \vec{r}_0)$. These two triangles divide the area of the paralellogram in two equal halves. Each of the two halves is equal to half the height H of the triangles times the length of the line segment. Formally:

$$\frac{1}{2}H.\,|\vec{r}_1 - \vec{r}_0| = \frac{1}{2}\,|(\vec{r}_1 - \vec{r}_0) \times (\vec{r} - \vec{r}_0)| \quad \Longrightarrow \quad H = \frac{|(\vec{r}_1 - \vec{r}_0) \times (\vec{r} - \vec{r}_0)|}{|\vec{r}_1 - \vec{r}_0|}$$

The height H of the triangle $(\vec{r}_0, \vec{r}_1, \vec{r})$ is precisely the length of the perpendicular from top $\vec{r}$ to base $(\vec{r}_0, \vec{r}_1)$. With other words, it is precisely the *distance* from a point (x, y) to the line segment $(x_0, y_0) - (x_1, y_1)$. Herewith we can write, less formally, for the fuzzyness of a straight line without the end-points:

$$L(x,y) \sim \frac{\text{thickness}}{\text{spread}} e^{-\frac{1}{2}(\text{distance}/\text{spread})^2}$$

Still another way of looking at the above is the following. First repeat:

$$B = \frac{|(x-x_0)(y_1-y_0)-(x_1-x_0)(y-y_0)|}{\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}}$$

The unit normal (n_x, n_y) of the line $(x-x_0)(y_1-y_0)-(x_1-x_0)(y-y_0) = 0$ is recognized herein:

$$\begin{bmatrix} n_x \\ n_y \end{bmatrix} = \frac{1}{\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}} \begin{bmatrix} +(y_1-y_0) \\ -(x_1-x_0) \end{bmatrix} = \begin{bmatrix} cos(\theta) \\ sin(\theta) \end{bmatrix}$$

Here θ is the angle of the *normal* of the straight line segment with the x-axis. The distance of a point (x, y) to the line $B = 0$ is precisely the length of the projection of $(x-x_0, y-y_0)$ on the normal $(cos(\theta), sin(\theta))$, which is just another (and easier) way to obtain the above results again. This results in a slightly different expression for the exponent in the exponential:

$$B = cos(\theta)(x-x_0) + sin(\theta)(y-y_0) \quad \Longrightarrow$$

$$L(x,y) = \frac{D}{\sigma\sqrt{2\pi}} \,.\, e^{-\frac{1}{2}[cos(\theta)(x-x_0)+sin(\theta)(y-y_0)]^2/\sigma^2}$$

So far so good. What will happen if we integrate the complete function $L(x, y)$, of the finite line segment, *with* the end-points, over the whole plane ? It is rather advantageous then to replace the infinitesimal volume $dx\,dy$ by another infinitesimal volume $ds\,dH$, where H is the distance to the line and s is the length of the arc measured along the line. The Jacobian determinant corresponding with this orthogonal coordinate transformation is just 1. And:

$$\iint L(x,y)\,dx\,dy = \iint L(s,H)\,ds\,dH =$$

$$\int_0^S \left[Erf(s) - Erf(s-S)\right]\,ds\,.\,\frac{D}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}H^2/\sigma^2}\,dH = S.D$$

That the first integral is indeed equal to S can be demonstrated with help of a suitable picture of the $Erf()$ functions. The outcome of the latter integral is well known from the theory of normal distributions. Anyway, the result is: length times width, which simply is the area (or number of pixels eventually) occupied by the line segment. A *normalized* line shall be defined with a thickness D equal to unity and a length S marginally extending to infinity. This is our end-result:

$$L(x,y) = \frac{1}{\sigma\sqrt{2\pi}}\,.\,e^{-\frac{1}{2}[cos(\theta)(x-x_0)+sin(\theta)(y-y_0)]^2/\sigma^2}$$

Sharpened Lines and Contours

It happens that the fuzzyfied equation of a straight line (segment) has to be integrated over all (black) pixels in its neighbourhood, resulting in a double integral, which is quite intensive to compute numerically:

$$\iint \frac{1}{\sigma\sqrt{2\pi}}\,.\,e^{-\frac{1}{2}[cos(\phi)(x-p)+sin(\phi)(y-p)]^2/\sigma^2}\,dx\,dy$$

A well known trick is to convert such an area integral into a line integral, the objective being to save an order of magnitude in computing time. This can be accomplished once again with help of Green's Theorem:

$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy =$$

Resulting indeed in the correct integrand for the area integral:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{\sigma\sqrt{2\pi}} \, . \, e^{-\frac{1}{2}[cos(\phi)(x-p)+sin(\phi)(y-q)]^2/\sigma^2} \left[cos^2(\phi) + sin^2(\phi)\right]$$

Because $cos^2(\phi) + sin^2(\phi) = 1$. Hence it may be concluded that the equivalent line integral is given by:

$$\oint (P\,dx + Q\,dy) =$$

$$\oint Erf\left[\frac{cos(\phi)(x-p) + sin(\phi)(y-q)}{\sigma}\right] [-sin(\phi)\,dx + cos(\phi)\,dy]$$

Where $[-sin(\phi)\,dx + cos(\phi)\,dy]$ denote increments along the contour, as they are (counter-clockwise) projected in the direction of the straight line. The last step is to take the limit of the above expression for $\sigma \to 0$. This results in an integral over a Heaviside function which is discontinuous just where the straight line is / a Heaviside which is jumping over the straight line - so to speak:

$$\oint H\,[cos(\phi)(x-p) + sin(\phi)(y-q)]\,[-sin(\phi)\,dx + cos(\phi)\,dy]$$

The outcome of this integral will be the length of the straight line, insofar as it is (covered by the area of interest, which in turn is) *enclosed* by the contours of the boundary integral $\oint$. The rest of the story should be considered again as a technology, not a theory. But I want to spend a few words on this one as well. The Heaviside function in the abovementioned integral implies again a *clipping problem*. But we are not finished with just one of these. As any straight line will eventually be restricted to a line *segment* with finite length, this will give rise to three subsequent clipping problems: one for the line segment itself and two for the lines perpendicular to the segment at its end points.

Disclaimers

Anything free comes without referee :-(
Succinctness is not my strongest quality.
My English may be better than your Dutch.