Fibonacci Iterations

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Consider the function:

$$f(x) = \frac{1}{x-1}$$

Suppose an initial value $x = x_0$ where $x_0 \neq 1$ has been given. Now form the iterates:

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(x_1) \\ x_3 &= f(x_2) \\ x_4 &= f(x_3) \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Or:

$$x_n = f(f(f(f(f(f(f(f(f(...f(x_0)))))..)))$$

Come on, let's just do it:

$$\begin{aligned} x_1 &= \frac{1}{x_0 - 1} \\ x_2 &= \frac{1}{x_1 - 1} = \frac{1}{\frac{1}{x_0-1} - 1} = \frac{x_0 - 1}{1 - (x_0 - 1)} = -\frac{x_0 - 1}{x_0 - 2} \\ x_3 &= \frac{1}{x_2 - 1} = \frac{1}{-\frac{x_0-1}{x_0-2} - 1} = -\frac{x_0 - 2}{(x_0 - 1) + (x_0 - 2)} = -\frac{x_0 - 2}{2x_0 - 3} \\ x_4 &= \frac{1}{x_3 - 1} = \frac{1}{-\frac{x_0-2}{2x_0-3} - 1} = -\frac{2x_0 - 3}{(x_0 - 2) + (2x_0 - 3)} = -\frac{x_0 - 2}{3x_0 - 5} \end{aligned}$$

See the pattern?

Theorem. The k'th iterand $(k \in Z, k > 0)$ is given by the fraction:

$$x_k = -\frac{F_{k-1}x_0 - F_k}{F_k x_0 - F_{k+1}}$$

Where F_k are terms of the Fibonacci Sequence $1, 1, 2, 3, 5, 8, 13, 21, ...$, which is defined (recursively) by:

$$\begin{aligned} F_k &= 0 \qquad &\text{for} \quad k \in Z, k < 0 \\ F_0 &= 1 \\ F_{k+1} &= F_k + F_{k-1} \qquad &\text{for} \quad k \in Z, k \geq 0 \end{aligned}$$

Proof. By induction to k. The theorem is true for $k = 1$. Assume that it is true for a certain $k = m$. Then calculate for $k = m + 1$:

$$x_{m+1} = \frac{1}{-\frac{F_{m-1}x_0 - F_m}{F_m x_0 - F_{m+1}} - 1} = -\frac{F_m x_0 - F_{m+1}}{(F_{m-1}x_0 - F_m) + (F_m x_0 - F_{m+1})}$$

$$= -\frac{F_m x_0 - F_{m+1}}{(F_{m-1} + F_m)x_0 - (F_m + F_{m+1})} = -\frac{F_{(m+1)-1}x_0 - F_{(m+1)}}{F_{(m+1)}x_0 - F_{(m+1)+1}}$$

Corollary. The iterations end up in *undefined* for all initial values (with $k \geq 0$):

$$x_0 = \frac{F_{k+1}}{F_k} \quad = \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \ldots$$

Because one of the denominators in the iterates becomes zero then. But if such is the case for x_k, then it's sensible to nevertheless continue the iterations with $x_{k+1} = 0$, because: $\lim_{x_k \to \pm\infty} 1/(x_k - 1) = 0$.

Theorem. If an iterand is given by $x_i = F_{k+1}/F_k$ then the next one is: $x_{i+1} = F_k/F_{k-1}$, provided that $(k \in Z, k > 1)$.

Proof.

$$x_{i+1} = \frac{1}{F_{k+1}/F_k - 1} = \frac{F_k}{F_{k+1} - F_k} = \frac{F_k}{F_{k-1}}$$

Corrolary. Thus the successive iterands are given by the sequence of so-called *Fibonacci Fractions*, in reverse order, i.e : 13/8 , 8/5 , 5/3 , 3/2 , 2/1 , 1/1 .

Lemma.

$$\lim_{k\to\infty} \frac{F_{k+1}}{F_k} = \phi \qquad \text{where} \quad \phi = \frac{1+\sqrt{5}}{2} \quad \text{is known as the } \textbf{golden ratio}$$

Proof.

$$\lim_{k\to\infty} \frac{F_{k+1}}{F_k} = \lim_{k\to\infty} \frac{F_k + F_{k-1}}{F_k} = 1 + \frac{1}{\lim_{k\to\infty} F_k/F_{k-1}}$$

If a limit exists, then $\lim_{k\to\infty} F_{k+1}/F_k$ and $\lim_{k\to\infty} F_k/F_{k-1}$ must be the same. Name it ϕ, then:

$$\phi = 1 + \frac{1}{\phi} \quad \Longrightarrow \quad \phi^2 - \phi - 1 = 0 \quad \Longrightarrow \quad \phi = \frac{1 \pm \sqrt{5}}{2}$$

Since it is clear that $\phi > 0$, only the solution $\phi = (1 + \sqrt{5})/2$ is valid.
Further details are omitted, for the reason that this is quite a well known result. The Scottish mathematician Robert Simson proved it in 1753:

`http://mathworld.wolfram.com/FibonacciNumber.html`

Theorem. There are two *invariant points* within the iterands, namely:

$$\psi := \frac{1-\sqrt{5}}{2} \quad \text{and} \quad \phi := \frac{1+\sqrt{5}}{2}$$

Proof. Solve the equation:

$$x = \frac{1}{x-1} \quad \Longrightarrow \quad x^2 - x - 1 = 0 \quad \Longrightarrow \quad x = \frac{1 \pm \sqrt{5}}{2}$$

Corrolary. Because they are roots of the same quadratic equation: $\phi.\psi = -1$ and $\phi + \psi = 1$.

Theorem. The sequence of iterands x_k always converges to $\psi = (1-\sqrt{5})/2$, which is irrespective of the initial value x_0. With one single exception, namely: $x_0 = (1+\sqrt{5})/2 = \phi$.

Proof. Use the lemma in:

$$\lim_{k\to\infty} -\frac{F_{k-1}x_0 - F_k}{F_k x_0 - F_{k+1}} = -\frac{\lim_{k\to\infty}(F_{k-1}/F_k)x_0 - 1}{x_0 - \lim_{k\to\infty}(F_{k+1}/F_k)} = -\frac{x_0/\phi - 1}{x_0 - \phi} = \frac{-1}{\phi}\,\frac{x_0-\phi}{x_0-\phi}$$

Here $-1/\phi = \psi$ and $(x_0 - \phi)/(x_0 - \phi) = 1$ except when $x_0 = \phi$.

Theorem. If the iterand x_k is negative, then x_{k+1} will be negative as well.

Proof.

$$x_k < 0 \quad \Longrightarrow \quad \frac{1}{x_k - 1} = -\frac{1}{|x_k| + 1} < 0$$

Corrolary. Once an iterand has become negative, all iterands that follow will remain negative.

Theorem. Only iterands x_k in the interval $1 < x_k < 2$ can result in x_{k+1} and x_{k+2} which are still positive.

Proof. An interand $x_k = 1$ results in undefined. If an iterand becomes $x_k < 1$ then the next x_{k+1} will be negative:

$$x_k < 1 \quad \Longrightarrow \quad x_{k+1} = \frac{1}{x_k - 1} < 0$$

An interand $x_k = 2$ results in undefined. If an iterand becomes $x_k > 2$ then the next x_{k+1} will be less than one (and thus x_{k+2} will be negative):

$$x_k > 2 \quad \Longrightarrow \quad x_k - 1 > 1 \quad \Longrightarrow \quad x_{k+1} = \frac{1}{x_k - 1} < 1$$

Corrolary. Thus our iterands can only remain positive in a rather *dangerous* interval, by which I mean that it is full of pitfalls like $\{\ 1/1\ ,\ 2/1\ ,\ 3/2\ ,\ 5/3\ \}$ eventually resulting in a division by zero. The behaviour of iterands *outside* the dangerous interval can be summarized as follows. You can jump in anywhere.
$(2 < x < \infty)\ \rightarrow\ (0 < x < 1)\ \rightarrow\ (-\infty < x < -1)\ \rightarrow\ (-1 < x < 0)$.
There is no escape from the latter interval, though:

Theorem. The abovementioned *invariant points* serve as a watershed:

$$\begin{array}{rcl} 1 < x_k < \phi & \Longrightarrow & \phi < x_{k+1} < +\infty \\ \phi < x_k < 2 & \Longrightarrow & 1 < x_{k+1} < \phi \\ -\infty < x_k < \psi & \Longrightarrow & \psi < x_{k+1} < 0 \\ \psi < x_k < 0 & \Longrightarrow & -1 < x_{k+1} < \psi \end{array}$$

Proof. Only the part with the square roots in it:

$$x_k - 1 <> \frac{1+\sqrt{5}}{2} - 1 = \frac{\sqrt{5}-1}{2} \quad \Longrightarrow \quad x_{k+1} >< \frac{1}{(\sqrt{5}-1)/2} = \frac{\sqrt{5}+1}{2}$$

$$x_k - 1 <> \frac{1-\sqrt{5}}{2} - 1 = -\frac{\sqrt{5}+1}{2} \quad \Longrightarrow \quad x_{k+1} >< -\frac{1}{(\sqrt{5}+1)/2} = \frac{1-\sqrt{5}}{2}$$

Corrolary. This means that successive iterands x_k show *oscillatory behaviour*. Especially in the neighbourhood of ψ, the point of convergence, they become, for example: $x_{10} < \psi\ ,\ x_{11} > \psi\ ,\ x_{12} < \psi\ ,\ x_{13} > \psi\ ....$.

Theorem. The invariant point $\psi = (1-\sqrt{5})/2$ is *stable*; the invariant point $\phi = (1+\sqrt{5})/2$ is *unstable*.

Proof. By *stable* we mean that small deviations become smaller and by *unstable* we mean that they become greater. Let's take a $\delta > 0$ and see what happens:

$$x_k = (1-\sqrt{5})/2 - \delta \quad \Longrightarrow \quad x_{k+1} = \frac{1}{-(1+\sqrt{5})/2 - \delta} = \frac{(1-\sqrt{5})/2}{1-\delta(1-\sqrt{5})/2}$$

Abbreviate: $\psi = (1-\sqrt{5})/2 \approx -0.618$. Then $(1-\delta.\psi) > 1$ and therefore:

$$x_{k+1} = \frac{\psi}{1-\delta.\psi} = \psi + \epsilon \quad \Longrightarrow \quad \epsilon = \frac{\psi - \psi(1-\delta.\psi)}{1-\delta.\psi} = \frac{\delta.\psi^2}{1-\delta.\psi} < 1$$

Abbreviate: $\phi = (1+\sqrt{5})/2 \approx +1.618$. Then $(1-\delta.\phi) < 1$ and therefore:

$$x_{k+1} = \frac{\phi}{1-\delta.\phi} = \phi + \epsilon \quad \Longrightarrow \quad \epsilon = \frac{\phi - \phi(1-\delta.\phi)}{1-\delta.\phi} = \frac{\delta.\phi^2}{1-\delta.\phi} > \delta.\phi^2 > 2.6\,\delta$$

Thus if $|x_k - \phi| = \delta$ then it follows that $|x_{k+1} - \phi| >> \delta$. QED .

Corrolary. The fact that ϕ is *unstable* explains its status as an exception. For it means that any disturbance in that value will explode as iterations proceed, thus finally resulting, nevertheless, in the proper limit, which is $\psi = (1-\sqrt{5})/2$. It is noticed, moreover, that ϕ is in that *dangerous* interval $1 < x_k < 2$ and that the pitfall values of F_{k+1}/F_k become very *dense* in the neighbourhood of ϕ .

Definition. The Fibonacci Fractions f_k for $k \in N, k \geq 0$ are: $f_k = F_{k+1}/F_k$. Some values are: $f_0 = 1/1$, $f_1 = 2/1$, $f_3 = 3/2$, $f_4 = 5/3$, $f_5 = 8/5$.

From previous theorems, it is clear that:

$$x_i = f_{k+1} \quad \Longleftrightarrow \quad x_{i+1} = f_k$$

$$x_i < \phi \quad \Longleftrightarrow \quad x_{i+1} > \phi \qquad \text{and} \qquad x_i > \phi \quad \Longleftrightarrow \quad x_{i+1} < \phi$$

Herewith we find: $f_0 < \phi$, $f_1 > \phi$, $f_2 < \phi$, $f_3 > \phi$, $f_4 < \phi$, $f_5 > \phi$, etcetera. Thus all *even* Fibonacci Fractions are *smaller* than the golden ratio while all *odd* Fibonacci Fractions are *greater* than the golden ratio. Meaning that every Fibonacci Fraction in the iterations sequence is surrounded either by two greater values either by two smaller values. With other words: the fractions f_k show *oscillatory* behaviour around the value of the golden ratio ϕ.

Lemma. A well known result is: $\psi^n = F_n - F_{n-1}\,\phi$ $(n \in N)$.

The lemma, but not the proof, is in the following nice reference on the web:

`http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/propsOfPhi.html`

Proof. The lemma is true for $n = 1$, because: $1/2 - \sqrt{5}/2 = 1 - 1(1/2 + \sqrt{5}/2)$. Now assume that the formula is true for $n = k$ and then prove it for $n = k+1$. Herewith we use the abovementioned properties $\phi + \psi = 1$ and $\phi.\psi = -1$.

$$\psi^k \psi = (F_k - F_{k-1}\,\phi)\psi = F_k(1-\phi) - F_{k-1}(-1) = (F_k + F_{k-1}) - F_k\,\phi$$

$$\Longrightarrow \quad \psi^{k+1} = F_{k+1} - F_k\,\phi$$

Corrolary. This can be written as: $f_k = F_{k+1}/F_k = \phi + \psi^{k+1}/F_k$. Here ψ^{k+1} is negative for *even* (k) and positive for *odd* (k) , which is entirely in concordance with the above result: the oscillatory behaviour of the Fibonacci Fractions. But it can be seen now, in addition, that the absolute values $|f_k - \phi|$ are *strongly decreasing* with increasing values of (k) . Convergence is extremely fast.

Theorem. If an iterand is given by $x_i < f_k$ then the next one is: $x_{i+1} > f_{k-1}$ and if an iterand is given by $x_i > f_k$ then the next one is: $x_{i+1} < f_{k-1}$.

Proof.

$$x_i < F_{k+1}/F_k \quad \Longrightarrow \quad x_{i+1} > \frac{1}{F_{k+1}/F_k - 1} = \frac{F_k}{F_{k-1}}$$
$$x_i > F_{k+1}/F_k \quad \Longrightarrow \quad x_{i+1} < \frac{1}{F_{k+1}/F_k - 1} = \frac{F_k}{F_{k-1}}$$

Corrolary. This gives us some idea how iterations proceed in the *dangerous* interval $1 < x < 2$. If an iterand x_i is somehow in between f_{k+1} and f_k (but not in between f_{k+2} and f_{k+1}), then the next iterand x_{i+1} will be somehow in between f_k and f_{k-1} (but not in between the previous f_{k+1} and f_k). And so on and so forth. The last possible iteration gives $1/1 < x < 3/2$. Then the iterands are "thrown out" of the dangerous interval: $1/(3/2-1) = 2$ and $1/(1-1) = \infty$, hence $2 < x < \infty$. The rest of the story is well known, as it has been outlined above: $(2 < x < \infty) \; \rightarrow \; (0 < x < 1) \; \rightarrow \; (-\infty < x < -1) \; \rightarrow \; (-1 < x < 0)$.

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch.