Quadrilateral Algebra

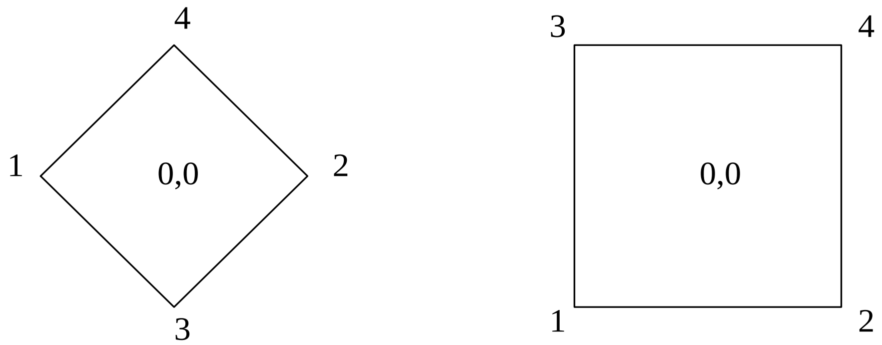
Let's consider the simplest finite element shape in two dimensions except one: the quadrilateral. Function behaviour is approximated inside a quadrilateral by a *bilinear* interpolation between the function values at the vertices or nodal points. Let T be such a function, and x, y coordinates. Then try:

$$T = A_T + B_T.x + C_T.y + D_T.x.y$$

Consider the quadrilateral as depicted in the picture below on the left. The vertex-coordinates of this quadrilateral are defined by the second and the third column of the matrix below. This matrix is formed by specifying T vertically for the nodal points and horizontally for the basic functions $1, x, y, x.y$:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 1 & +\frac{1}{2} & 0 & 0 \\ 1 & 0 & -\frac{1}{2} & 0 \\ 1 & 0 & +\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} A_T \\ B_T \\ C_T \\ D_T \end{bmatrix}$$

The last column of the matrix is zero. Hence it is *singular*, meaning that A, B, C and D cannot be found in this manner. It turns out that such a method, though it has been employed successfully for a triangle, cannot be used for an element which is interpolated by a function other than a linear one.



A solution can be found, however. Assume that the same expression is valid for the function T as well as for the coordinates x and y. Herewith it is expressed that we have, again, an *isoparametric transformation*. The next step is then to find a suitable *parent-element*. Our first attempt didn't't work out, but maybe we are more lucky with the element depicted on the right in the same figure. The interpolation function looks alike, though it is expressed now in *local coordinates* ξ and η :

$$T = A_T + B_T.\xi + C_T.\eta + D_T.\xi.\eta$$

It is assumed that the local coordinates inside a parent quadrilateral are between $-\frac{1}{2} \le \xi \le +\frac{1}{2}$ and $-\frac{1}{2} \le \eta \le +\frac{1}{2}$, meaning that the parent quad is actually a square. Now specify again for the vertices and the basic functions:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & +\frac{1}{4} \\ 1 & +\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} \\ 1 & -\frac{1}{2} & +\frac{1}{2} & -\frac{1}{4} \\ 1 & +\frac{1}{2} & +\frac{1}{2} & +\frac{1}{4} \end{bmatrix} \begin{bmatrix} A_T \\ B_T \\ C_T \\ D_T \end{bmatrix} \qquad \text{F.E} \leftarrow \text{F.D.}$$

It is remarked that the above matrix is *orthogonal*, i.e. its columns are mutually perpendicular. This also means that the "condition" of the matrix is optimal. In fact it's even better. Apart from scaling factors, the inverse matrix is equal to the transposed, which can be determined easily:

$$\begin{bmatrix} A_T \\ B_T \\ C_T \\ D_T \end{bmatrix} = \begin{bmatrix} +\frac{1}{4} & +\frac{1}{4} & +\frac{1}{4} & +\frac{1}{4} \\ -\frac{1}{2} & +\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} & +\frac{1}{2} \\ +1 & -1 & -1 & +1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} \qquad \text{F.D} \leftarrow \text{F.E.}$$

Writing out the matrix notation:

$$\begin{aligned} A_T &= \tfrac{1}{4}(+T_1 + T_2 + T_3 + T_4) \\ B_T &= \tfrac{1}{2}(-T_1 + T_2 - T_3 + T_4) \\ C_T &= \tfrac{1}{2}(-T_1 - T_2 + T_3 + T_4) \\ D_T &= 1(+T_1 - T_2 - T_3 + T_4) \end{aligned}$$

Hence A_T, B_T, C_T, D_T are equal to local partial derivatives:

$$T(0) = A_T \quad ; \quad \frac{\partial T}{\partial \xi}(0) = B_T \quad ; \quad \frac{\partial T}{\partial \eta}(0) = C_T \quad ; \quad \frac{\partial^2 T}{\partial \xi \partial \eta} = D_T$$

These coefficients form a Finite Difference formulation:

$$T = T(0) + \frac{\partial T}{\partial \xi}(0).\xi + \frac{\partial T}{\partial \eta}(0).\eta + \frac{\partial^2 T}{\partial \xi \partial \eta}.\xi.\eta$$

Shape functions may be constructed as follows:

$$\begin{bmatrix} 1 & \xi & \eta & \xi.\eta \end{bmatrix} \begin{bmatrix} +\frac{1}{4} & +\frac{1}{4} & +\frac{1}{4} & +\frac{1}{4} \\ -\frac{1}{2} & +\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} & +\frac{1}{2} \\ +1 & -1 & -1 & +1 \end{bmatrix}$$

When written out as:

$$\begin{aligned} N_1 &= \tfrac{1}{4} - \tfrac{1}{2}\xi - \tfrac{1}{2}\eta + \xi.\eta = (\tfrac{1}{2} - \xi).(\tfrac{1}{2} - \eta) \\ N_2 &= \tfrac{1}{4} + \tfrac{1}{2}\xi - \tfrac{1}{2}\eta - \xi.\eta = (\tfrac{1}{2} + \xi).(\tfrac{1}{2} - \eta) \\ N_3 &= \tfrac{1}{4} - \tfrac{1}{2}\xi + \tfrac{1}{2}\eta - \xi.\eta = (\tfrac{1}{2} - \xi).(\tfrac{1}{2} + \eta) \\ N_4 &= \tfrac{1}{4} + \tfrac{1}{2}\xi + \tfrac{1}{2}\eta + \xi.\eta = (\tfrac{1}{2} + \xi).(\tfrac{1}{2} + \eta) \end{aligned}$$

these coefficients form a Finite Element formulation:

$$T = N_1.T_1 + N_2.T_2 + N_3.T_3 + N_4.T_4$$

Any shape function N_k has a value 1 at vertex (k) and it is zero at all other vertices. Global and local coordinates of an arbitrary quadrilateral are related to each other via the isoparametric transformation:

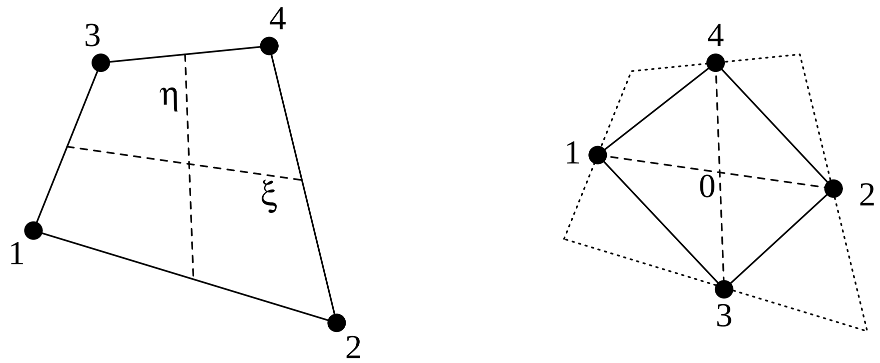
$$\begin{aligned} x &= N_1.x_1 + N_2.x_2 + N_3.x_3 + N_4.x_4 \\ y &= N_1.y_1 + N_2.y_2 + N_3.y_3 + N_4.y_4 \end{aligned}$$

The equivalent Finite Difference representation is:

$$\begin{aligned} x(\xi,\eta) &= A_x + B_x.\xi + C_x.\eta + D_x.\xi.\eta \\ y(\xi,\eta) &= A_y + B_y.\xi + C_y.\eta + D_y.\xi.\eta \end{aligned} \qquad \text{where:}$$

$$\begin{array}{lcl} A_x = \frac{1}{4}(x_1 + x_2 + x_3 + x_4) & ; & A_y = \frac{1}{4}(y_1 + y_2 + y_3 + y_4) \\ B_x = \frac{1}{2}(x_2 + x_4) - \frac{1}{2}(x_1 + x_3) & ; & B_y = \frac{1}{2}(y_2 + y_4) - \frac{1}{2}(y_1 + y_3) \\ C_x = \frac{1}{2}(x_3 + x_4) - \frac{1}{2}(x_1 + x_2) & ; & C_y = \frac{1}{2}(y_3 + y_4) - \frac{1}{2}(y_1 + y_2) \\ D_x = (+x_1 - x_2 - x_3 + x_4) & ; & D_y = (+y_1 - y_2 - y_3 + y_4) \end{array}$$

The origin of the local (ξ,η) coordinate system is determined by $\xi = 0$ and $\eta = 0$. Hence by $(x,y)(O) = (A_x, A_y) = (\overline{x}, \overline{y})$ = midpoint = centre of gravity. The ξ-axis is defined by $-\frac{1}{2} < \xi < +\frac{1}{2}$ and $\eta = 0$. Hence by the (dashed) line $(x,y) = (A_x, A_y) + \xi.(B_x, B_y)$.
The η-axis is defined by $-\frac{1}{2} < \eta < +\frac{1}{2}$ and $\xi = 0$. Hence by the (dashed) line $(x,y) = (A_x, A_y) + \eta.(C_x, C_y)$.



Function behaviour at the sides of a bi-linear quadrilateral is *linear*:

$$\begin{aligned} T(-\tfrac{1}{2},\eta) &= (\tfrac{1}{2} - \eta).T_1 + (\tfrac{1}{2} + \eta).T_3 \\ T(+\tfrac{1}{2},\eta) &= (\tfrac{1}{2} - \eta).T_2 + (\tfrac{1}{2} + \eta).T_4 \\ T(\xi,-\tfrac{1}{2}) &= (\tfrac{1}{2} - \xi).T_1 + (\tfrac{1}{2} + \xi).T_2 \\ T(\xi,+\tfrac{1}{2}) &= (\tfrac{1}{2} - \xi).T_3 + (\tfrac{1}{2} + \xi).T_4 \end{aligned}$$

Meaning that the function values at the midpoints of the sides are the average of function values at the vertices. Anyway, it is trivial that:

$$\frac{1}{2}(T_1 + T_2) + \frac{1}{2}(T_3 + T_4) = \frac{1}{2}(T_1 + T_3) + \frac{1}{2}(T_2 + T_4)$$

Now suppose that inside the existing quad a new quadrilateral is constructed, also with vertices which are numbered according to $(1, 2, 3, 4)$. A transition from the old to the new vertices may be defined as follows (see figure):

$$\begin{array}{ll} \frac{1}{2}(T_1 + T_3) \Rightarrow T_1 & \frac{1}{2}(T_2 + T_4) \Rightarrow T_2 \\ \frac{1}{2}(T_1 + T_2) \Rightarrow T_3 & \frac{1}{2}(T_3 + T_4) \Rightarrow T_4 \end{array}$$

For the values of the function T at the vertices of the internal quad we find the following, according to the above triviality for average values:

$$T_1 + T_2 = T_3 + T_4$$

This is valid also for the global coordinates x and y, due to isoparametrics, though it can also be derived directly with some knowledge of planar geometry:

$$x_1 + x_2 = x_3 + x_4 \qquad y_1 + y_2 = y_3 + y_4$$

The geometrical meaning of this being that the internal quadrilateral is always *a parallelogram*.
The Finite Difference representation of the original quadrilateral may be used for deriving an F.D. representation for the internal quad:

$$T = A_T + B_T.\xi + C_T.\eta + D_T.\xi.\eta$$

Rewritten to the new vertices:

$$\begin{array}{l} A_T = \frac{1}{4}(T_1 + T_2 + T_3 + T_4) = \frac{1}{2}(T_1 + T_2) = \frac{1}{2}(T_3 + T_4) = T_0 \\ B_T = T_2 - T_1 = \partial T/\partial \xi \\ C_T = T_4 - T_3 = \partial T/\partial \eta \\ D_T = 0 \end{array}$$

The term A_T is just the average value which is localised at the midpoint of the original quad, or the local origin, as $T(0)$; we write T_0.
From the fact that the term with D_T is zero, the following may be concluded:

By joining the midpoints of the sides of an arbitrary quadrilateral, the so-called *internal* quadrilateral of the original quad may be constructed. No restriction of generality is the following equation, which imposes an extra relationship upon the nodal values of an arbitrary function T at the internal quadrilateral element:

$$T_1 + T_2 = T_3 + T_4$$

Not only at triangles, but also at the internal element of an arbitrary quadrilateral, function behaviour appears to be *linear*:

$$T = T_0 + (T_2 - T_1).\xi + (T_4 - T_3).\eta$$

For the global coordinates we can write, in the same way:

$$\begin{aligned} x &= x_0 + (x_2 - x_1).\xi + (x_4 - x_3).\eta \\ y &= y_0 + (y_2 - y_1).\xi + (y_4 - y_3).\eta \end{aligned}$$

Due to these linear relationships, it is possible to express ξ and η vice versa in x and y:

$$\begin{aligned} \xi &= [+(y_4 - y_3)(x - x_0) - (x_4 - x_3)(y - y_0)]/\Delta \\ \eta &= [-(y_2 - y_1)(x - x_0) + (x_2 - x_1)(y - y_0)]/\Delta \end{aligned}$$

With:

$$\Delta = (x_2 - x_1).(y_4 - y_3) - (x_4 - x_3).(y_2 - y_1)$$

An interpretation of (ξ, η) as area-coordinates, like with the linear triangle, appears to be feasible. Because any functional relationship is in fact linear, we can also write:

$$\begin{aligned} T_2 - T_1 &= \partial T/\partial x.(x_2 - x_1) + \partial T/\partial y.(y_2 - y_1) \\ T_4 - T_3 &= \partial T/\partial x.(x_4 - x_3) + \partial T/\partial y.(y_4 - y_3) \end{aligned}$$

The inverse of this problem reads:

$$\begin{aligned} \partial T/\partial x &= [(T_2 - T_1).(y_4 - y_3) - (T_4 - T_3).(y_2 - y_1)]/\Delta \\ \partial T/\partial y &= [(x_2 - x_1).(T_4 - T_3) - (x_4 - x_3).(T_2 - T_1)]/\Delta \end{aligned}$$

Giving for the internal element of a quadrilateral:

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \begin{bmatrix} -(y_4 - y_3) & +(y_4 - y_3) & +(y_2 - y_1) & -(y_2 - y_1) \\ +(x_4 - x_3) & -(x_4 - x_3) & -(x_2 - x_1) & +(x_2 - x_1) \end{bmatrix} /\Delta$$

Herewith it is said that the gradient-operator, at an internal quadrilateral, is represented by a 2×4 *Differentiation Matrix*. Let us repeat, at last:

$$\begin{aligned} A_T &= \tfrac{1}{2}(T_1 + T_2) = \tfrac{1}{2}(T_3 + T_4) \\ B_T &= T_2 - T_1 \\ C_T &= T_4 - T_3 \end{aligned}$$

Four equations with four unknowns, which can hence be solved:

$$\begin{aligned} T_1 &= A_T - \tfrac{1}{2}B_T \\ T_2 &= A_T + \tfrac{1}{2}B_T \\ T_3 &= A_T - \tfrac{1}{2}C_T \\ T_4 &= A_T + \tfrac{1}{2}C_T \end{aligned}$$

This is recognized as the irreversible matrix that we found at the beginning of this chapter. The circle is closed by noting that the parent-element of the internal quadrilateral has been depicted already in the first figure on the left.