Re: Why exp(-st) in the Laplace Transform?

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Operator Calculus

Operator Calculus is a mathematical technique which is employed in physics (i.e. quantum mechanics), without any serious concern about its validity. When it comes to theoretical mathematics, however, the same technique meets extremely cautious formulations. It almost seems as if Operator Calculus is like kind of an illegal activity, for the "real" mathematician at least.
Operators can be seen as mathematical devices that operate on functions. Hence, actually, operators are functions of functions. Concerning the kind of operations that are involved, *differentiation* and *integration* will be considered in the first place.
Let ψ be a function. Then the operation (d/dx) in $d\psi/dx$ is an example of a (differential) operator. But also multiplication with another function is an operator, such as (f) in $f\psi$. Now we have the following obvious definitions for equality, sums and products of operators α, β and (soon also) γ, when applied on arbitrary functions ψ and ϕ :

$$[\,\alpha = \beta\,] \;\equiv\; [\,\alpha\psi = \beta\psi\,] \quad ; \quad (\alpha+\beta)\psi \;\equiv\; \alpha\psi + \beta\psi \quad ; \quad (\alpha\beta)\psi \;\equiv\; \alpha(\beta\psi)$$

An operator is called *linear* if the following two requirements are fulfilled. Here let λ be a scalar. Strictly speaking, the second requirement could have been derived from the first:

$$\alpha(\psi+\phi) = \alpha\psi + \alpha\phi \quad ; \quad \alpha(\lambda\psi) = \lambda(\alpha\psi)$$

In the sequel, all of our utterances will be restricted to linear operators. It is a simple excercise to prove the following Rules of Arithmetic:

$$\alpha+\beta = \beta+\alpha \quad ; \quad (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$$

$$(\alpha\beta)\gamma = \alpha(\beta\gamma) \quad ; \quad (\alpha+\beta)\gamma = \alpha\gamma+\beta\gamma$$

$$\gamma(\alpha+\beta) = \gamma\alpha+\gamma\beta \quad ; \quad \alpha\lambda = \lambda\alpha$$

The rules for manipulating linear operators are indeed very much resemblant to the arithmetic rules for ordinary numbers. With *one* single exception. And this is actually the *only* thing one should keep in mind, in practice, when performing arithmetic with (linear) operators. The commutative law, namely, is NOT valid:

$$\alpha\beta \neq \beta\alpha$$

For this reason, the *commutator* of two operators is defined as:

$$[\alpha, \beta] = \alpha\beta - \beta\alpha$$

The commutator of two operators, in general, will not be zero. Furthermore we define an inverse and a (repeatedly) composite operator:

$$[\,\beta = \alpha^{-1}\,] \;\equiv\; [\,\alpha\beta = 1\,] \quad ; \quad \alpha^n \;\equiv\; \alpha...\alpha \text{ (n terms)}$$

Let's become somewhat more specific. The product rule for differentiation reads: $(f.\psi)' = f'.\psi + f.\psi'$. Or:

$$\frac{d}{dx}(f.\psi) = (\frac{df}{dx})\psi + (f.\frac{d}{dx})\psi$$

The function ψ, being entirely arbitrary, provides not a shred of information. Therefore it would be desirable to leave it out. Working conditions which enable us to do so have been created by the above operator-definitions. It is quite clear, namely, that it is always possible to arrive at an expression of the form: $\alpha\psi = \beta\psi$, which makes it possible to leave out ψ. Yet this is, for true mathematicians, a tender spot: because a single d/dx cannot possibly "mean anything". Mainstream mathematics obviously has some problems with a deed of *real* abstraction, like this one:

$$\frac{d}{dx}f = \frac{df}{dx} + f.\frac{d}{dx}$$

The (non-commutative) law for composing a differential and a product-operator is derived herefrom:

$$\left[\frac{d}{dx}, f\right] = \frac{d}{dx}f - f\frac{d}{dx} = \frac{df}{dx}$$

After division by f the earlier formula can also be written as folllows:

$$f^{-1}\frac{d}{dx}f = \frac{d}{dx} + \frac{f'}{f}$$

The fraction f'/f reminds of the derivative of $log(f)$. If we put $f'/f = g$ then $log(f) = \int g\, dx$, hence $f = exp(\int g\, dx)$. Let's replace the name g by the name f again. At last, exchange the left and the right side. Then the end-result is:

$$\boxed{\frac{d}{dx} + f = e^{-\int f\, dx}\,\frac{d}{dx}\,e^{+\int f\, dx}}$$

It is expected from the reader that he or she transfers this formula to his or her non-volatile memory, so to speak. It is an extremely useful result, namely, as will be demonstrated now at hand of three examples.

Differential Equations

Operator Calculus can be applied for the purpose of finding solutions of ordinary linear Differential Equations. Three examples will be given.

Example 1

Solve an ordinary second order linear differential equation in $y(x)$ with constant coefficients:

$$ay'' + by' + cy = 0$$

Operator Calculus enables us to abstract as much as possible from the solution $y(x)$ itself:

$$\left[a \left(\frac{d}{dx} \right)^2 + b \, \frac{d}{dx} + c \right] y(x) = 0$$

What we are going to do next is, don't be surprised: *decompose into factors*. We are (almost) forced to do so, because the operator (d/dx) and the constants a, b and c mutually behave *as if* they were ordinary numbers: the commutator of a differentiation and a constant is zero. Remember that the commutative law for (linear) operators is the only thing which could be deviant from common algebra with ordinary numbers.

$$a \left(\frac{d}{dx} \right)^2 + b \left(\frac{d}{dx} \right) + c = \left(\frac{d}{dx} - \lambda_1 \right) \left(\frac{d}{dx} - \lambda_2 \right)$$

Where λ_1 and λ_2 are roots of the so-called *characteristic equation*:

$$a\lambda^2 + b\lambda + c = 0$$

Herewith, the differential equation can be rewritten as:

$$\left(\frac{d}{dx} - \lambda_1 \right) \left(\frac{d}{dx} - \lambda_2 \right) y(x) = 0$$

We are going to employ now the "extremely useful formula" from the previous secion:

$$\frac{d}{dx} - \lambda_{1,2} = e^{-\int -\lambda_{1,2} \, dx} \, \frac{d}{dx} \, e^{+\int -\lambda_{1,2} \, dx}$$
$$= e^{\lambda_{1,2} x} \, \frac{d}{dx} \, e^{-\lambda_{1,2} x}$$

Giving, at last, for the O.D.E. the equivalent expression:

$$e^{\lambda_1 x} \frac{d}{dx} \, e^{-\lambda_1 x} e^{\lambda_2 x} \frac{d}{dx} \, e^{-\lambda_2 x} y(x) = 0$$

Systematic integration is possible now:

$$e^{-\lambda_1 x} e^{\lambda_2 x} \frac{d}{dx} \, e^{-\lambda_2 x} y(x) = C_1 \quad ; \quad e^{-\lambda_2 x} y(x) = C_1 \int e^{(\lambda_1 - \lambda_2)x} \, dx$$

As has been said, λ can be solved from $a\lambda^2 + b\lambda + c = 0$, a quadratic equation with discriminant: $D = b^2 - 4ac$. Two different cases are to be distinguished.
(a) $\lambda_1 \neq \lambda_2$:

$$\int e^{(\lambda_1 - \lambda_2)x} \, dx = \frac{e^{(\lambda_1 - \lambda_2)x}}{\lambda_1 - \lambda_2} + C_2$$

Giving as a solution:

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \qquad C_1, C_2 \text{ arbitrary}$$

(b) $\lambda_1 = \lambda_2$:

$$\int dx = x$$

Giving as a solution:

$$y(x) = e^{\lambda x}[C_1.x + C_2] \qquad C_1, C_2 \text{ arbitrary}$$

The above derivation provides a sharp contrast with the heuristics in official documents about differential equations. What makes it so special is the fact that this method leads to the solution in a completely natural way. It is not necessary, at all, to make some kind of miraculous assumption about the nature of the solution. In particular, there is nothing mysterious about the special case $\lambda_1 = \lambda_2$. Nothing comes out of the blue sky.

Example 2

Solve the differential equation by Euler in $y(x)$, a and b constant:

$$x^2 y'' + axy' + by = 0$$

Again, we will abstract as much as possible from the solution $y(x)$:

$$\left[x^2 \left(\frac{d}{dx} \right)^2 + \, ax \, \frac{d}{dx} + b \right] y(x) = 0$$

Employ the commutator $[d/dx, x] = 1$ for changing $x.d/dx$ into $d/dx.x - 1$ and herewith $x.(x.d/dx).d/dx$ in $x.d/dx.x.d/dx - x.d/dx$. This is necessary for rewriting the O.D.E. a little bit, namely as follows:

$$\left[\left(x \frac{d}{dx} \right)^2 + (a-1) \left(x \frac{d}{dx} \right) + b \right] y = 0$$

Let's try to decompose into factors:

$$\left(x\frac{d}{dx}\right)^2 + (a-1)\left(x\frac{d}{dx}\right) + b = \left(x\frac{d}{dx} - \lambda_1\right)\left(x\frac{d}{dx} - \lambda_2\right)$$

Where λ_1 and λ_2 are roots of the characteristic equation:

$$\lambda^2 + (a-1)\lambda + b = 0$$

Herewith, the differential equation by Euler can be rewritten as:

$$\left(x\frac{d}{dx} - \lambda_1\right)\left(x\frac{d}{dx} - \lambda_2\right) y(x) = 0$$

Or:

$$x\left(\frac{d}{dx} - \frac{\lambda_1}{x}\right) x\left(\frac{d}{dx} - \frac{\lambda_2}{x}\right) y(x) = 0$$

We are going to employ again the formula, as memorized from the previous section:

$$\frac{d}{dx} - \frac{\lambda_{1,2}}{x} = e^{-\int -\frac{\lambda_{1,2}}{x}\,dx}\ \frac{d}{dx}\ e^{+\int -\frac{\lambda_{1,2}}{x}\,dx}$$

$$= e^{\lambda_{1,2}log(x)}\ \frac{d}{dx}\ e^{-\lambda_{1,2}log(x)} = x^{\lambda_{1,2}}\frac{d}{dx}\ x^{-\lambda_{1,2}}$$

Giving for the O.D.E. in its final form:

$$x.x^{\lambda_1}\frac{d}{dx}\ x^{-\lambda_1}x.x^{\lambda_2}\frac{d}{dx}\ x^{-\lambda_2}y(x) = 0$$

Systematic integration is possible now:

$$x^{-\lambda_1}x.x^{\lambda_2}\frac{d}{dx}\ x^{-\lambda_2}y(x) = C_1 \quad ; \quad x^{-\lambda_2}y(x) = C_1\int x^{\lambda_1-\lambda_2-1}\,dx$$

As has been said, λ can be solved from $\lambda^2+(a-1)\lambda+b=0$, a quadratic equation with discriminant: $D = (a-1)^2 - 4b$. Two different cases are distinguished.
(a) $\lambda_1 \neq \lambda_2$:

$$\int x^{\lambda_1-\lambda_2-1}\,dx = \frac{x^{\lambda_1-\lambda_2}}{\lambda_1-\lambda_2} + C_2$$

Giving as a solution:

$$y(x) = C_1x^{\lambda_1} + C_2x^{\lambda_2} \qquad C_1, C_2 \text{ arbitrary}$$

(b) $\lambda_1 = \lambda_2$:

$$\int x^{-1}\,dx = log(x)$$

Giving as a solution:

$$y(x) = x^{\lambda}[C_1.log(x) + C_2] \qquad C_1, C_2 \text{ arbitrary}$$

Again, it is not necessary to make special assumptions about the shape of the solution. The result is obtained in a completely natural way, for the general as well as for the special case.

Example 3

As a last example, we solve the following differential equation:

$$r\frac{d^2p}{dr^2} + (2 - v_0 r)\frac{dp}{dr} - 2v_0 p = 0$$

Here: p = unknown function (kind of pressure), r = radial distance (to the sun), v_0 = scaled velocity (of the solar wind). This equation arises with the simplification of a far more complicated problem: a mathematical model for calculating the anomalous component of cosmic radiation in the heliosphere. So far so good. Decompose into factors:

$$r\left(\frac{d}{dr} + \frac{2}{r}\right)\left(\frac{d}{dr} - v_0\right)p = 0$$

Use the Basic Formula:

$$r\frac{1}{r^2}\frac{d}{dr}r^2\, e^{v_0 r}\frac{d}{dr}e^{-v_0 r}p = 0$$

Systematical integration gives:

$$\frac{d}{dr}e^{-v_0 r}p = C_1\frac{e^{-v_0 r}}{r^2}$$

$$p(r) = C_1 e^{v_0 r}\int\frac{e^{-v_0 r}}{r^2}dr + C_2 e^{v_0 r}$$

Thus we have the following elementary solutions:

$$p_1(r) = e^{v_0 r}\int\frac{e^{-v_0 r}}{r^2}dr$$

$$p_2(r) = e^{v_0 r}$$

The integral can be worked out further with help of partial integration. Let $t = -v_0 r$ in:

$$\int\frac{e^t}{t^2}dt = -\frac{e^t}{t} + \int\frac{e^t}{t}dt$$

Herewith for the first solution:

$$p_1(r) = \frac{1}{v_0 r} + e^{v_0 r}.Ei(-v_0 r)$$

Here Ei is the so-called *exponential integral*, for which it is known that it cannot be written in a closed form:

$$Ei(x) = \int_{-\infty}^{x}\frac{e^t}{t}dt$$

For those who don't believe, the outcome can always be verified with help of a Computer Algebra System. We used the CAS package MAPLE for doing this:

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p:=1/(v0*r)+exp(v0*r)*Ei(-v0*r);
r*diff(diff(p,r),r)+(2-v0*r)*diff(p,r)-2*v0*p;
simplify(");
quit;
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Unnecessary to remark that the outcome, indeed, becomes zero, as expected.

Laplace and Statistics

It is known from quantum mechanics that the law of conservation of momentum can be derived from translation symmetry. If it is possible to move around the physical system in space without its properties being altered, then in that case conservation of momentum is guaranteed. However, the foundation of this theorem is purely mathematical, and can be understood as follows. The series expansion of a function $f(x+\xi)$ around x is:

$$f(x+\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{k!} f^{(k)}(x) = \left[\sum_{k=0}^{\infty} \frac{1}{k!} \left(\xi \, \frac{d}{dx} \right)^k \right] f(x)$$

In the expression between square brackets the series expansion of the exponential function e^x is recognized. Therefore we can write, symbolically:

$$f(x+\xi) = e^{\xi \frac{d}{dx}} f(x)$$

With this formula in mind, consider an arbitrary convolution-integral:

$$\int_{-\infty}^{+\infty} h(\xi)\phi(x-\xi)\, d\xi$$

Convolution integrals do frequently occur. With a linear system, the response at a disturbance is the convolution-integral of the disturbance with the so-called unity-response. The unity-response is the way in which the system reacts upon the simplest of all disturbances, that is a steep peak of very short duration at time zero, a "delta-function". Convolution integrals can be rewritten with help of the operator-expression for $f(x-\xi)$ as follows:

$$\int_{-\infty}^{+\infty} h(\xi) e^{-\xi \frac{d}{dx}}\, d\xi \quad \phi(x)$$

The integral in this expression should be well known to us. Quite "by incidence", namely, it is just the (double-sided) Laplace transformation:

$$H(p) = \int_{-\infty}^{+\infty} h(\xi) e^{-\xi p}\, d\xi$$

It seems that Laplace's integral shows up quite spontaneously with elementary considerations about convolution-integrals in combination with our Operator Calculus. The formula for the convolution-integral can now be written as:

$$H(\frac{d}{dx})\,\phi(x)$$

The fact that Laplace transforms are a powerful means for solving differential equations can now be understood without much effort. Suppose we have a linear inhomogeneous differential equation. In general it has the form:

$$D(\frac{d}{dx})\,\phi(x) = f(x)$$

Then with help of Operator Calculus we can immediately write the solution as:

$$\phi(x) = \frac{1}{D(\frac{d}{dx})} f(x)$$

Put $H(d/dx) = 1/D(d/dx)$, then the excercise becomes: find the inverse Laplace transform of $H(p)$. Let this inverse function be called $h(x)$. Finding the solution then follows entirely the abovementioned pattern:

$$\phi(x) = \int_{-\infty}^{+\infty} h(\xi) f(x-\xi)\, d\xi$$

So far so good. Investigate the Laplace transform of $exp(-\mu t^2)$:

$$\int_{-\infty}^{+\infty} e^{-pt} e^{-\mu t^2}\, dt = \int_{-\infty}^{+\infty} e^{-\mu x^2}\, dx . e^{p^2/4\mu} = \sqrt{\frac{\pi}{\mu}} e^{p^2/4\mu}$$

with help of: $x = t + p/2\mu$. Laplace transform H and inverse Laplace transform h are mutually related as follows, after replacing $1/4\mu$ by $1/2\sigma^2$:

$$H(p) = e^{\frac{1}{2}\sigma^2 p^2} \quad \Longleftrightarrow \quad h(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma^2}$$

A convolution integral with the normal distribution $h(t)$ as a kernel can thus be written, with the help of Operator Calculus, as:

$$\int_{-\infty}^{+\infty} h(\xi)\phi(x-\xi)\, d\xi = e^{\frac{1}{2}\sigma^2 \frac{d^2}{dx^2}} \phi(x)$$

The physical meaning of this is that the operator $exp(\frac{1}{2}\sigma^2 \frac{d^2}{dx^2})$ "smears out" the function $\phi(x)$ over domains with size of the order σ.
So-called "moment generating functions" play a role in Statistics. They are the expectation values of the exponential function $exp(pt)$; with other words: they

are Laplace transforms of probability densities. In a handbook about Statistics it is read as follows, again:

$$M(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] e^{tx}\, dx =$$

$$\frac{e^{\mu t+\frac{1}{2}\sigma^2 t^2}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} exp\left[\frac{-(x-\mu-\sigma^2 t)^2}{2\sigma^2}\right]\, dx = e^{\mu t+\frac{1}{2}\sigma^2 t^2}$$

The corresponding normal "fuzzying"-operator is in general given by:

$$M(\frac{d}{dx}) = e^{\mu\frac{d}{dx}+\frac{1}{2}\sigma^2\frac{d^2}{dx^2}} = e^{\mu\frac{d}{dx}}\, e^{\frac{1}{2}\sigma^2\frac{d^2}{dx^2}}$$

The outcome is immediately applicable to the following problem. Consider the (partial differential)equation for diffusion of heat in space and time:

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}$$

Rewrite in the first place:

$$\lambda\frac{\partial}{\partial t}T = \lambda a\frac{\partial^2}{\partial x^2}T$$

As a next step we exponentiate at both sides the operator in place:

$$e^{\lambda\partial/\partial t}\, T = e^{\lambda a\partial^2/\partial x^2}\, T$$

At both sides are operator-expressions which can be converted into classical mathematics with the acquired knowledge:

$$T(x, t+\lambda) = \int_{-\infty}^{+\infty} h(\xi)T(x-\xi, t)\, d\xi$$

Where $\frac{1}{2}\sigma^2 = \lambda a$. Therefore:

$$h(t) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{\,-t^2/2\sigma^2}$$

Now exchange t and λ, and substitute $\lambda = 0$. Then we find, while travelling via a very short route, the solution of the equation for thermal diffusion:

$$T(x,t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi at}}\, e^{\,-\xi^2/(4at)}\, T(x-\xi, 0)\, d\xi$$

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch.