Simplicial Moments simply at a Triangle

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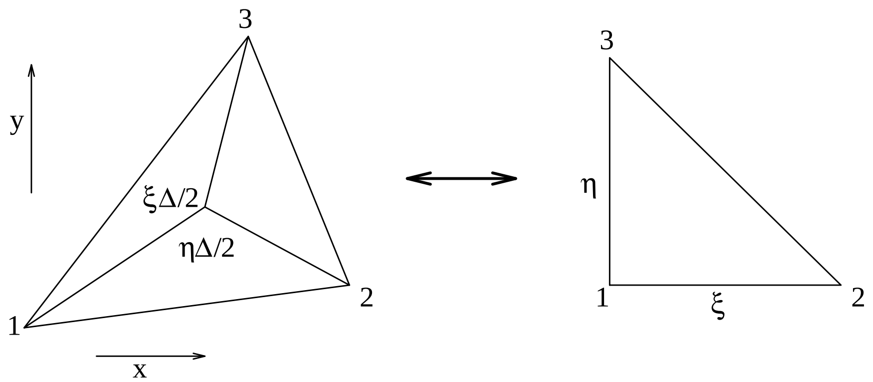
Triangle Algebra

Let's consider the simplest non-trivial finite element shape in two dimensions: the linear triangle. Function behaviour is approximated inside such a triangle by a *linear* interpolation between the function values at the vertices, also called: nodal points. Let T be such a function, and x, y coordinates, then:

$$T = A.x + B.y + C$$

Where the constants A, B, C are yet to be determined.



Substitute $x = x_k$ and $y = y_k$ with $k = 1, 2, 3$:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} C \\ A \\ B \end{bmatrix}$$

The first of these equations can already be used to eliminate the constant C, once and forever:

$$T_1 = A.x_1 + B.y_1 + C$$

Resulting in:

$$T - T_1 = A.(x - x_1) + B.(y - y_1)$$

Hence the constants A and B are determined by:

$$\begin{aligned} T_2 - T_1 &= A.(x_2 - x_1) + B.(y_2 - y_1) \\ T_3 - T_1 &= A.(x_3 - x_1) + B.(y_3 - y_1) \end{aligned}$$

Two equations with two unknowns. The solution is found by straightforward elimination, or by applying Cramer's rule:

$$\begin{aligned} A &= [(y_3 - y_1).(T_2 - T_1) - (y_2 - y_1).(T_3 - T_1)]/\Delta \\ B &= [(x_2 - x_1).(T_3 - T_1) - (x_3 - x_1).(T_2 - T_1)]/\Delta \end{aligned}$$

There are several forms of the determinant Δ, which should be memorized when it is appropriate:

$$\begin{aligned}
\Delta &= (x_2 - x_1).(y_3 - y_1) - (x_3 - x_1).(y_2 - y_1) \\
\Delta &= 2 \times \text{ area of triangle} \\
\Delta &= x_1.y_2 + x_2.y_3 + x_3.y_1 - y_1.x_2 - y_2.x_3 - y_3.x_1 \\
\Delta &= x_1.(y_2 - y_3) + x_2.(y_3 - y_1) + x_3.(y_1 - y_2) \\
\Delta &= y_1.(x_3 - x_2) + y_2.(x_1 - x_3) + y_3.(x_2 - x_1) \\
\Delta &= \left\| \begin{array}{ccc} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{array} \right\|
\end{aligned}$$

Anyway, it is concluded that:

$$T - T_1 = \xi.(T_2 - T_1) + \eta.(T_3 - T_1)$$

Where:

$$\begin{array}{l} \xi = [(y_3 - y_1).(x - x_1) - (x_3 - x_1).(y - y_1)]/\Delta \\ \eta = [(x_2 - x_1).(y - y_1) - (y_2 - y_1).(x - x_1)]/\Delta \end{array}$$

Or:

$$\left[\begin{array}{c} \xi \\ \eta \end{array} \right] = \left[\begin{array}{cc} +(y_3 - y_1) & -(x_3 - x_1) \\ -(y_2 - y_1) & +(x_2 - x_1) \end{array} \right] / \Delta \left[\begin{array}{c} x - x_1 \\ y - y_1 \end{array} \right]$$

The inverse of the following problem is recognized herein:

$$\left[\begin{array}{c} x - x_1 \\ y - y_1 \end{array} \right] = \left[\begin{array}{cc} (x_2 - x_1) & (x_3 - x_1) \\ (y_2 - y_1) & (y_3 - y_1) \end{array} \right] \left[\begin{array}{c} \xi \\ \eta \end{array} \right]$$

Or:

$$\begin{array}{l} x - x_1 = \xi.(x_2 - x_1) + \eta.(x_3 - x_1) \\ y - y_1 = \xi.(y_2 - y_1) + \eta.(y_3 - y_1) \end{array}$$

But also:

$$T - T_1 = \xi.(T_2 - T_1) + \eta.(T_3 - T_1)$$

Therefore the *same* expression holds for the function T as well as for the coordinates x and y . This is precisely what people mean by an *isoparametric* ("same parameters") transformation, a terminology which is quite common in Finite Element contexts. Now recall the formulas which express ξ and η into x and y :

$$\begin{array}{l} \xi = [(y_3 - y_1).(x - x_1) - (x_3 - x_1).(y - y_1)]/\Delta \\ \eta = [(x_2 - x_1).(y - y_1) - (y_2 - y_1).(x - x_1)]/\Delta \end{array}$$

Thus ξ can be interpreted as: area of the sub-triangle spanned by the vectors $(x - x_1, y - y_1)$ and $(x_3 - x_1, y_3 - y_1)$ divided by the whole triangle area. And η can be interpreted as: area of the sub-triangle spanned by the vectors $(x - x_1, y - y_1)$

and $(x_2 - x_1, y_2 - y_1)$ divided by the whole triangle area. This is the reason why ξ and η are sometimes called *area-coordinates*; see the above figure, where (two times) the area of the triangle as a whole is denoted as Δ. There exist even *three* of these coordinates in literature. But the third area-coordinate is, of course, dependent on the other two, being equal to $(1 - \xi - \eta)$. Instead of area-coordinates, we prefer to talk about *local coordinates* ξ and η of an element, in contrast to the *global coordinates* x and y. It is possible that local coordinates coincide with the global coordinates. A triangle for which such is the case is called a *parent element*. The portrait of the parent triangle is also depicted in the above figure: it is rectangular, and two sides of it are equal.
Let's reconsider the expression:

$$T - T_1 = \xi.(T_2 - T_1) + \eta.(T_3 - T_1)$$

Partial differentiation to ξ and η gives:

$$\partial T/\partial\xi = T_2 - T_1 \quad ; \quad \partial T/\partial\eta = T_3 - T_1$$

Therefore, with node (1) as the origin, hence $T(0) = T_1$:

$$T = T(0) + \xi\frac{\partial T}{\partial\xi} + \eta\frac{\partial T}{\partial\eta}$$

This is part of a Taylor series expansion around node (1). Such Taylor series expansions are quite common in Finite Difference analysis. Now rewrite as follows:

$$T = (1 - \xi - \eta).T_1 + \xi.T_2 + \eta.T_3$$

Here the functions $(1 - \xi - \eta)$, ξ, η are called the *shape functions* of the Finite Element. Shape functions N_k have the property that they are unity in one of the nodes (k), and zero in all other nodes. In our case:

$$N_1 = 1 - \xi - \eta \quad ; \quad N_2 = \xi \quad ; \quad N_3 = \eta$$

So we have two representations, which are almost trivially equivalent:

$$\begin{array}{ll} T = T_1 + \xi.(T_2 - T_1) + \eta.(T_3 - T_1) & : \text{Finite Difference like} \\ T = (1 - \xi - \eta).T_1 + \xi.T_2 + \eta.T_3 & : \text{Finite Element like} \end{array}$$

What kind of terms can be discretized at the domain of a linear triangle? In the first place, the function $T(x, y)$ itself, of course. But one may also try the first order partial derivatives $\partial T/\partial x$, $\partial T/\partial y$. We find:

$$\begin{array}{l} \partial T/\partial x = A = [(y_3 - y_1).(T_2 - T_1) - (y_2 - y_1).(T_3 - T_1)]/\Delta \\ \partial T/\partial y = B = [(x_2 - x_1).(T_3 - T_1) - (x_3 - x_1).(T_2 - T_1)]/\Delta \end{array}$$

By collecting terms belonging to the same T_k, this can also be written as:

$$\Delta \begin{bmatrix} \partial T/\partial x \\ \partial T/\partial y \end{bmatrix} = \begin{bmatrix} +(y_2 - y_3) & +(y_3 - y_1) & +(y_1 - y_2) \\ -(x_2 - x_3) & -(x_3 - x_1) & -(x_1 - x_2) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

Or, in operator form:

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} / \Delta$$

The right hand side will be called a *Differentiation Matrix* in subsequent work. Thus the gradient operator at any linear triangle is represented by a 2×3 differentiation matrix.

Triangle Integrals

Our goal is to calculate some (first, second, third order) moments of an arbitrary triangle. To be mathematically precise:

$$\mu_x = \iint x \, dxdy \quad \text{and} \quad \mu_y = \iint y \, dxdy$$

$$\sigma_{xx} = \iint x^2 \, dxdy \quad \text{and} \quad \sigma_{yy} = \iint y^2 \, dxdy \quad \text{and} \quad \sigma_{xy} = \iint xy \, dxdy$$

Or even:

$$M_{xxx} = \iint x^3 \, dxdy \quad \text{and} \quad M_{xxy} = \iint x^2 y \, dxdy$$

$$M_{xyy} = \iint xy^2 \, dxdy \quad \text{and} \quad M_{yyy} = \iint y^3 \, dxdy$$

Following the theory in the previous paragraph, global coordinates (x, y) can be expressed in their local counterparts (ξ, η) :

$$x - x_0 = \xi.(x_1 - x_0) + \eta.(x_2 - x_0)$$
$$y - y_0 = \xi.(y_1 - y_0) + \eta.(y_2 - y_0)$$

It makes no difference for the outcome of the integrals if a more handsome choice for the coordinate system is to be preferred. Therefore, let one of the vertices of the triangle, say (x_0, y_0), be selected as the origin of our global coordinate system. Then:

$$x = \xi.x_1 + \eta.x_2 \quad \text{and} \quad y = \xi.y_1 + \eta.y_2$$

And the Jacobian Δ of this transformation is involved with:

$$dx.dy = (x_1 y_2 - x_2 y_1) \, d\xi d\eta = \Delta \, d\xi d\eta$$

Limited use will be made of Newton's binomial formula:

$$(a+b)^n = \sum_{k=0}^{n} C(n,k)\, a^k b^{n-k} = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!}\, a^k b^{n-k}$$

The formulas for any moment of a triangle take the following general form:

$$\iint x^m y^n \, dxdy = \iint (\xi.x_1 + \eta.x_2)^m (\xi.y_1 + \eta.y_2)^n \, \Delta \, d\xi d\eta =$$

$$\iint \sum_{i=0}^{m} C(m,i)\, \xi^i x_1^i \eta^{m-i} x_2^{m-i} \sum_{j=0}^{n} C(n,j)\, \xi^j y_1^j \eta^{n-j} y_2^{n-j} \, \Delta \, d\xi d\eta =$$

$$\Delta \sum_{i=0}^{m} \sum_{j=0}^{n} C(m,i)\, x_1^i x_2^{m-i} C(n,j)\, y_1^j y_2^{n-j} \iint \xi^{i+j} \eta^{m+n-i-j} \, d\xi d\eta$$

The above formula - being far too complicated - will not be used in the sequel. It turns out, however, that we have to calculate integrals like:

$$F(m,n) = \iint \xi^m \eta^n \, d\xi d\eta$$

Herewith the integration is carried out over a rectangular equilateral triangle, with local coordinates ξ and η, where $0 \le \xi \le 1$ and $0 \le \eta \le (1-\xi)$. Working out a few steps:

$$F(m,n) = \iint \xi^m \eta^n \, d\xi d\eta = \int_0^1 \xi^m \left[\int_0^{1-\xi} \eta^n d\eta \right] d\xi$$

$$= \int_0^1 \xi^m \left[\frac{(1-\xi)^{n+1}}{n+1} \right] d\xi = \int_0^1 \frac{(1-\xi)^{n+1}}{n+1} d\left(\frac{\xi^{m+1}}{m+1} \right)$$

$$= \left[\frac{(1-\xi)^{n+1}}{n+1} \frac{\xi^{m+1}}{m+1} \right]_0^1 - \int_0^1 \frac{\xi^{m+1}}{m+1} d\left(\frac{(1-\xi)^{n+1}}{n+1} \right) = 0 + \int_0^1 \frac{\xi^{m+1}}{m+1} (1-\xi)^n \, d\xi$$

$$= \frac{n}{m+1} \int_0^1 \xi^{m+1} \left[\frac{(1-\xi)^n}{n} \right] d\xi = \frac{n}{m+1} \int_0^1 \xi^{m+1} \left[\int_0^{1-\xi} \eta^{n-1} d\eta \right] d\xi$$

$$= \frac{n}{m+1} \iint \xi^{m+1} \eta^{n-1} d\xi d\eta = \frac{n}{m+1} F(m+1, n-1)$$

Now we can set up the following sequence of formulas:

$$F(m,n) = \frac{n}{m+1} F(m+1,n-1) = \frac{n}{m+1} \frac{n-1}{m+2} F(m+2,n-2) = ...$$

$$= \frac{n(n-1)\ldots 2.1}{(m+1)(m+2)\ldots(m+n-1)(m+n)} F(m+n,0)$$

$$= \frac{n(n-1)\ldots 2.1\,.\,m(m-1)\,..\,2.1}{1.2.\ldots(m-1)m(m+1)\ldots(m+n)} F(m+n,0) = \frac{m!\,n!}{(m+n)!} F(m+n,0)$$

So only integrals of the form $F(m+n,0)$ are left to be calculated:

$$\iint \xi^{m+n}\, d\xi d\eta = \int_0^1 \xi^{m+n} \left[\int_0^{1-\xi} d\eta \right] d\xi = \int_0^1 \xi^{m+n}(1-\xi) d\xi$$

$$= \int_0^1 \xi^{m+n} d\xi - \int_0^1 \xi^{m+n+1} d\xi = \left[\frac{\xi^{m+n+1}}{m+n+1} \right]_0^1 - \left[\frac{\xi^{m+n+2}}{m+n+2} \right]_0^1$$

$$= \frac{1}{m+n+1} - \frac{1}{m+n+2} = \frac{(m+n+2)-(m+n+1)}{(m+n+1)(m+n+2)}$$

$$= \frac{1}{(m+n+1)(m+n+2)} = F(m+n,0)$$

Hence:

$$F(m,n) = \frac{m!\,n!}{(m+n)!} \frac{1}{(m+n+1)(m+n+2)} = \frac{m!\,n!}{(m+n+2)!}$$

This is the final result:

$$\iint \xi^m \eta^n \, d\xi d\eta = \frac{m!\,n!}{(m+n+2)!}$$

Now let's calculate a few of these triangle moments.

$$\text{Area} = \iint dxdy = \iint d\xi d\eta\, \Delta = \frac{0!\,0!}{(0+0+2)!}\, \Delta = \Delta/2 = \frac{1}{2}(x_1 y_2 - x_2 y_1)$$

Since all (other) moments have to be divided by this area, the outcome of their integrals have to be multiplied with a factor $2/\Delta$. A first order moment is:

$$\frac{\iint x\, dxdy}{\text{Area}} = 2/\Delta \iint (x_1\xi + x_2\eta)\, d\xi d\eta\, \Delta = 2x_1 \iint \xi\, d\xi d\eta + 2x_2 \iint \eta\, d\xi d\eta$$

$$= 2x_1 \frac{1!\,0!}{(1+0+2)!} + 2x_2 \frac{0!\,1!}{(0+1+2)!} = x_1\, 2/6 + x_2\, 2/6 = (x_1+x_2)/3$$

In very much the same way (replace x by y) we can prove that:

$$\frac{\iint y\, dxdy}{\text{Area}} = (y_1+y_2)/3$$

Different though it seems, this is the same as the familiar result that the coordinates of the midpoint of a triangle equal one-third of the coordinates of the vertices:

$$\overline{x} = x_0 + 1/3\,[(x_1-x_0)+(x_2-x_0)] = (x_0+x_1+x_2)/3$$
$$\overline{y} = y_0 + 1/3\,[(y_1-y_0)+(y_2-y_0)] = (y_0+y_1+y_2)/3$$

Second order moments are:

$$2/\Delta \iint x^2 \, dxdy \quad \text{and} \quad 2/\Delta \iint y^2 \, dxdy \quad \text{and} \quad 2/\Delta \iint xy \, dxdy$$

It is sufficient to calculate only the last integral. Proper substitutions in $\overline{xy}$ will take care of the other two later on.

$$2/\Delta \iint xy \, dxdy = 2 \iint (x_1\xi + x_2\eta)(y_1\xi + y_2\eta) \, d\xi d\eta$$

$$= 2x_1y_1 \iint \xi^2 \, d\xi d\eta + 2x_1y_2 \iint \xi\eta \, d\xi d\eta + 2x_2y_1 \iint \xi\eta \, d\xi d\eta + 2x_2y_2 \iint \eta^2 \, d\xi d\eta$$

$$= 2x_1y_1 \frac{2!\,0!}{(2+0+2)!} + 2x_1y_2 \frac{1!\,1!}{(1+1+2)!} + 2x_2y_1 \frac{1!\,1!}{(1+1+2)!} + 2x_2y_2 \frac{0!\,2!}{(0+2+2)!}$$

$$= 2x_1y_1.1/12 \; + \; 2x_1y_2.1/24 \; + \; 2x_2y_1.1/24 \; + \; 2x_2y_2.1/12$$

The result is:

$$\overline{xy} = (2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2)/12$$

Substitute x instead of y herein:

$$\overline{xx} = (2x_1x_1 + x_1x_2 + x_2x_1 + 2x_2x_2)/12 \quad \Longrightarrow \quad \overline{x^2} = (x_1x_1 + x_1x_2 + x_2x_2)/6$$

Or the reverse: y instead of x. Giving:

$$\overline{y^2} = (y_1y_1 + y_1y_2 + y_2y_2)/6$$

However, second order moments should be evaluated preferrably with respect to the midpoint:

$$\overline{xy} - \overline{x}\,\overline{y} = (2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2)/12 - (x_1 + x_2)/3.(y_1 + y_2)/3$$

$$= (6x_1y_1 + 3x_1y_2 + 3x_2y_1 + 6x_2y_2 - 4x_1x_1 - 4x_1y_2 - 4x_2y_1 - 4x_2y_2)/36$$

$$= (2x_1x_1 - x_1y_2 - x_2y_1 + 2x_2y_2)/36$$

By proper substitution:

$$\overline{x^2} - \overline{x}^2 = (x_1x_1 - x_1x_2 + x_2x_2)/18 \quad \text{and} \quad \overline{y^2} - \overline{y}^2 = (y_1y_1 - y_1y_2 + y_2y_2)/18$$

Attention is restricted to the moment in the x-direction, because the one in the y-direction is quite analogous.

$$\overline{x^2} - \overline{x}^2 = (x_1^2 - x_1x_2 + x_2^2)/18 = \left[x_1^2 + (x_1^2 - 2x_1x_2 + x_2^2) + x_2^2\right]/36$$

$$= \left[x_1^2 + (x_1 - x_2)^2 + x_2^2\right]/36$$

More lucid expressions are obtained by introducing a non-zero origin again:

$$\begin{aligned}
\overline{x^2} - \overline{x}^2 &= \left[(x_1 - x_0)^2 + (x_1 - x_2)^2 + (x_2 - x_0)^2\right]/36 \\
\overline{y^2} - \overline{y}^2 &= \left[(y_1 - y_0)^2 + (y_1 - y_2)^2 + (y_2 - y_0)^2\right]/36
\end{aligned}$$

The question remains to be answered why these triangle integrals are supposed to be so important. Well, virtually every domain in the plane can be thought as being built up from (small) triangles. This means that every integral over such a domain can effectively be thought as a (weighted) sum of integrals over nothing else but triangles (k):

$$\frac{\iint x^m y^n \, dxdy}{\iint dxdy} = \frac{\sum_k \left[\iint x^m y^n \, dxdy\right]_k}{\sum_k \left[\iint dxdy\right]_k} = \frac{\sum_k \left[\iint x^m y^n \, d\xi d\eta \, \Delta\right]_k}{\sum_k \left[\iint d\xi d\eta \, \Delta\right]_k}$$

$$= \sum_k \left[\iint x^m y^n \, d\xi d\eta\right]_k . \, w_k \qquad \text{where} \quad w_k = \frac{\Delta_k}{\frac{1}{2}\sum_i \Delta_i}$$

That is: the triangle moments are weighted with their individual areas, divided by the total area of the domain. This is the main reason why triangle moments are so useful: you can compose all other planar moments out of them (within certain accuracy bounds). Provided that you are not too punctilious, nothing else will be needed.