Streamfunction and Potential

In the 'HdBstu .pas' source code, the ow eld around a circular cylinder is modelled, which obeys the following system of (Ideal Flow) equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad ; \quad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

Here: x = coordinate in horizontal direction and y = coordinate in vertical direction, u = component of velocity in x-direction and v = component of velocity in y-direction.
It should have been demonstrated in a previous chapter that the "natural" place of velocity vectors is at the midside nodes of quadrilateral elements. In the 'HdBstu .pas' code, the Ideal Flow velocity eld is rst calculated "ab initio". Then the Streamfunction and Potential can both be derived from the ow eld, by employing the equations:

$$d\phi = u.dx + v.dy \quad ; \quad d\psi = u.dy - v.dx$$

These are discretized by integrating over a nite line segment $(1,2)$, giving:

$$\phi_2 - \phi_1 = u.(x_2 - x_1) + v.(y_2 - y_1) \quad ; \quad \psi_2 - \psi_1 = u.(y_2 - y_1) - v.(x_2 - x_1)$$

In order to be able to apply the reverse procedure, *both* the Potential and the Streamfunction must be calculated rst. For the Ideal Flow problem at hand, this can be done by solving standard Laplace equations numerically, with help of a standard Finite Element procedure. The boundary conditions of these problems are yet to be de ned.
After solving the Laplace problem for *both* ϕ and ψ, nally the velocity eld can be calculated, using the above in a reverse fashion, namely as two equations with two unknowns:

$$(x_2 - x_1).u + (y_2 - y_1).v = \phi_2 - \phi_1$$
$$(y_2 - y_1).u - (x_2 - x_1).v = \psi_2 - \psi_1$$

The solution (u, v) of this system is:

$$u = \frac{(\psi_2 - \psi_1)(y_2 - y_1) + (\phi_2 - \phi_1)(x_2 - x_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$v = \frac{(\phi_2 - \phi_1)(y_2 - y_1) - (\psi_2 - \psi_1)(x_2 - x_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

It is remarked that the denominator in these expressions is always positive and that the problem of "numerical di erentiation" is very well conditioned here, because the accompanying matrix (apart from minor details) is orthogonal:

$$\begin{bmatrix} (x_2 - x_1) & (y_2 - y_1) \\ (y_2 - y_1) & -(x_2 - x_1) \end{bmatrix}$$