Streamlines

In the "NeraCalc" program, a flow field with cylinder symmetry is calculated, which obeys the following system of (Ideal Flow) equations:

$$\frac{@r{:}u}{@r} + \frac{@r{:}v}{@z} = 0 \qquad ; \qquad \frac{@u}{@z} - \frac{@v}{@r} = 0$$

Here: r = coordinate in radial direction and z = coordinate in axial direction, u = component of velocity in radial direction and v = component of velocity in axial direction.
A streamline in this flow field is characterized by the fact that the velocity vector is tangent in every point:

$$\frac{dz}{dr} = \frac{v}{u}$$

Suppose that there exists a function   in such a way that the streamlines are coincident with the isolines of $(r;z)$. The differential of any such function can be written as:

$$d \quad = \frac{@}{@r}dr + \frac{@}{@z}dz$$

An isoline of   is characterized by the fact that the function does not change when travelling along this line: $d \quad = 0$.
As a consequence, along an isoline, the following holds:

$$\frac{@}{@r}dr + \frac{@}{@z}dz = 0 \qquad \Longrightarrow \qquad \frac{dz}{dr} = -\frac{@ \quad =@r}{@ \quad =@z}$$

$$\text{But:} \qquad \frac{dz}{dr} = \frac{v}{u} \qquad \text{Hence:} \qquad -\frac{@ \quad =@r}{@ \quad =@z} = \frac{v}{u}$$

From which it may be concluded that:

$$\frac{@}{@r} = \quad {:}v \qquad ; \qquad \frac{@}{@z} = - \quad {:}u$$

Where the constant   is yet to be determined.

It is required, in addition, that the function   is a true differentiable and smooth function. To that end, the following equation must hold:

$$\frac{@}{@r}\left(\frac{@}{@z}\right) = \frac{@}{@z}\left(\frac{@}{@r}\right)$$

Hence:

$$\frac{@(- \quad {:}u)}{@r} = \frac{@ \quad {:}v}{@z} \qquad \Longrightarrow \qquad \frac{@ \quad {:}u}{@r} + \frac{@ \quad {:}v}{@z} = 0$$

Now the flow field, whether Ideal or not, is characterized by the fact that the equation of continuity (conservation of mass) is valid:

$$\frac{@r{:}u}{@r} + \frac{@r{:}v}{@z} = 0$$

It is concluded, therefore, that a sufficient condition is obtained by: $\quad = r$. From which it is inferred that:

$$\frac{@}{@r} = r{:}v \quad ; \quad \frac{@}{@z} = -r{:}u \qquad \Longrightarrow \qquad d \quad = r{:}v{:}dr - r{:}u{:}dz$$

Let's integrate this expression over a certain finite distance, characterized by endpoints $(1; 2)$:

$$\int_1^2 d \quad = \int_1^2 r{:}v{:}dr - \int_1^2 r{:}u{:}dz$$

The radial coordinate r as well as the velocities u and v will vary only linearly over this small distance, thus assuming mean values:

$$_2 - \quad _1 = \frac{1}{2}(r_2^2 - r_1^2)v - \frac{1}{2}(r_1 + r_2)(z_2 - z_1)u$$

An expression which may be simplified as follows:

$$_2 - \quad _1 = \frac{1}{2}(r_2 + r_1)\left[(r_2 - r_1){:}v - (z_2 - z_1){:}u\right]$$

This is the final discretization formula, as has been used in the "NeraCalc" program, for calculating the streamline function called .