Fuzzyfied Lissajous Analysis

Contours are obtained routinely from the processing of black and white images. This chapter is concerned with topics on *Lissajous Figures*, especially those obtained by Fourier Analysis of closed curves or contours. This results in the insight that every such closed curve is approximated by a superposition of *Lissajous Ellipses* and simple formulas can be derived for zero'th, first and second order momenta, in terms of the Fourier coefficients alone.

Lissajous Series

Consider a contour in the plane, consisting of a pair of coordinate functions $(x, y)(t)$, where t is the running parameter. It is an important fact that any such contour is a closed curve without self-intersection: a Jordan curve. Because the contour is closed, each of the functions $x(t)$ and $y(t)$ can be considered as being *periodic* with period L, where L is the length of just one walk around the contour. Consequently, the functions $x(t)$ and $y(t)$ each can be developed into a Fourier series:

$$x(t) = \frac{1}{2}A_x(0) + \sum_{k=1}^{\infty}\left[A_x(k)\,cos\left(k\frac{2\pi}{L}t\right) + B_x(k)\,sin\left(k\frac{2\pi}{L}t\right)\right]$$

$$y(t) = \frac{1}{2}A_y(0) + \sum_{k=1}^{\infty}\left[A_y(k)\,cos\left(k\frac{2\pi}{L}t\right) + B_y(k)\,sin\left(k\frac{2\pi}{L}t\right)\right]$$

To be taken together as:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A_x(0) \\ A_y(0) \end{bmatrix} + \sum_{k=1}^{\infty}\left\{\begin{bmatrix} A_x(k) \\ A_y(k) \end{bmatrix} cos\left(k\frac{2\pi}{L}t\right) + \begin{bmatrix} B_x(k) \\ B_y(k) \end{bmatrix} sin\left(k\frac{2\pi}{L}t\right)\right\}$$

Where the *Fourier coefficients* are to be evaluated, according to:

$$\begin{bmatrix} A_x(k) \\ A_y(k) \end{bmatrix} = \frac{1}{\frac{1}{2}L}\int_0^L \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} cos\left(k\frac{2\pi}{L}t\right)\,dt$$

$$\begin{bmatrix} B_x(k) \\ B_y(k) \end{bmatrix} = \frac{1}{\frac{1}{2}L}\int_0^L \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} sin\left(k\frac{2\pi}{L}t\right)\,dt$$

It is remarked that this Fourier series is quite analogous - if not to say: merely equal - to a superposition of *Lissajous figures*. These Lissajous figures share the common midpoint $\frac{1}{2}\left[A_x(0), A_y(0)\right]$. The lowest (except zero) order figure is an ellipse (perhaps a degenerated one. Meaning, for example, that a circle can be described completely with just *one* more term of the series.)
With Pattern Recognition in pictures, contours are obtained in a discrete form. Therefore it is advantageous to evaluate the integrals of the Fourier/Lissajous coefficients numerically. Because the discrete coordinates of the contour points

are obtained as midpoints of pixel boundaries, midpoint integration preferrably may be employed:

$$\begin{bmatrix} A_x(k) \\ A_y(k) \end{bmatrix} = \frac{1}{\frac{1}{2}L} \sum_{m=1}^{L} \begin{bmatrix} x_m \\ y_m \end{bmatrix} cos\left(k\frac{2\pi}{L}m\right)$$

$$\begin{bmatrix} B_x(k) \\ B_y(k) \end{bmatrix} = \frac{1}{\frac{1}{2}L} \sum_{m=1}^{L} \begin{bmatrix} x_m \\ y_m \end{bmatrix} sin\left(k\frac{2\pi}{L}m\right)$$

Where it is remarked that the dt increments are equal to 1 .
With a Lissajous series expansion, the *differential geometry* of a contour is an easy job. Set $\omega = 2\pi/L$ and start with:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} A_x(0) \\ A_y(0) \end{bmatrix} + \sum_{k=1}^{\infty} \left\{ \begin{bmatrix} A_x(k) \\ A_y(k) \end{bmatrix} cos(k\omega t) + \begin{bmatrix} B_x(k) \\ B_y(k) \end{bmatrix} sin(k\omega t) \right\} \quad \Longrightarrow$$

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \sum_{k=1}^{\infty} \left\{ \begin{bmatrix} A_x(k) \\ A_y(k) \end{bmatrix} (-k\omega) sin(k\omega t) + \begin{bmatrix} B_x(k) \\ B_y(k) \end{bmatrix} (+k\omega) cos(k\omega t) \right\}$$

It looks like if, for $k > 0$:

$$\begin{bmatrix} A'_x(k) \\ A'_y(k) \end{bmatrix} = k\omega \begin{bmatrix} B_x(k) \\ B_y(k) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B'_x(k) \\ B'_y(k) \end{bmatrix} = -k\omega \begin{bmatrix} A_x(k) \\ A_y(k) \end{bmatrix}$$

Suggesting a recursive method like:

$$\begin{bmatrix} A_x^{(n+1)}(k) \\ A_y^{(n+1)}(k) \end{bmatrix} = k\omega \begin{bmatrix} B_x^{(n)}(k) \\ B_y^{(n)}(k) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B_x^{(n+1)}(k) \\ B_y^{(n+1)}(k) \end{bmatrix} = -k\omega \begin{bmatrix} A_x^{(n)}(k) \\ A_y^{(n)}(k) \end{bmatrix}$$

Where:

$$\begin{bmatrix} B_x^{(0)}(k) \\ B_y^{(0)}(k) \end{bmatrix} = \begin{bmatrix} B_x(k) \\ B_y(k) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_x^{(0)}(k) \\ A_y^{(0)}(k) \end{bmatrix} = \begin{bmatrix} A_x(k) \\ A_y(k) \end{bmatrix}$$

Fuzzyfied Lissajous

Summation instead of integration will be valid, as long as the discretization interval is much smaller than the period of the sine and cosine functions in the series, which is L/k. It will be demonstrated now that this condition is met in a very natural way if *fuzzyfied* contours are considered instead:

$$\begin{bmatrix} \overline{x}(t) \\ \overline{y}(t) \end{bmatrix} = \int_{-\infty}^{+\infty} \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{1}{2}(t-\tau)^2/\sigma^2} d\tau$$

(not only) In this case, it's easier to work with complex Fourier coefficients:

$$\begin{bmatrix} \overline{c}_x(k) \\ \overline{c}_y(k) \end{bmatrix} = \frac{1}{L} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} \begin{bmatrix} \overline{x}(t) \\ \overline{y}(t) \end{bmatrix} e^{i\,k\omega\,t} \, dt$$

Where $\omega = 2\pi/L$. Exchange the integral signs and look what happens:

$$\begin{bmatrix} \overline{c}_x(k) \\ \overline{c}_y(k) \end{bmatrix} = \frac{1}{L} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} \left\{ \int_{-\infty}^{+\infty} \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{1}{2}(t-\tau)^2/\sigma^2} d\tau \right\} e^{i\,k\omega\,t} \, dt =$$

$$\frac{1}{L} \int_{-\infty}^{+\infty} \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} \left\{ \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{1}{2}(t-\tau)^2/\sigma^2} e^{i\,k\omega\,t} \, dt \right\} d\tau$$

Concentrate on the exponential functions:

$$e^{-\frac{1}{2}(t-\tau)^2/\sigma^2} e^{i\,k\omega\,t} = e^{-\frac{1}{2}(t-\tau)^2/\sigma^2 + i\,k\omega\,t}$$

Where:

$$-\frac{1}{2}(t-\tau)^2/\sigma^2 + i\,k\omega\,t = -\frac{1}{2\sigma^2}\left[(t-\tau)^2 - 2\sigma^2 i\,k\omega\,t\right] =$$

$$-\frac{1}{2\sigma^2}\left[(t-\tau)^2 - 2\sigma^2 i\,k\omega\,(t-\tau) + \left(\sigma^2 i\,k\omega\right)^2\right] + i\,k\omega\,\tau + \frac{1}{2\sigma^2}\left(\sigma^2 i\,k\omega\right)^2 =$$

$$-\frac{1}{2\sigma^2}\left[(t-\tau) - \sigma^2 i\,k\omega\right]^2 + i\,k\omega\,\tau - \frac{1}{2}\left(\sigma k\omega\right)^2$$

Hence:

$$e^{-\frac{1}{2}(t-\tau)^2/\sigma^2} e^{i\,k\omega\,t} = e^{-\frac{1}{2}\left[(t-\tau)-\sigma^2 i\,k\omega\right]^2/\sigma^2} e^{i\,k\omega\,\tau} e^{-\frac{1}{2}(\sigma k\omega)^2}$$

Giving $[\overline{c}_x(k), \overline{c}_y(k)] =$

$$e^{-\frac{1}{2}(\sigma k\omega)^2} \frac{1}{L} \int_{-\infty}^{+\infty} \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} \left\{ \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{1}{2}\left[t-(\tau+\sigma^2 i\,k\omega)\right]^2/\sigma^2} dt \right\} e^{i\,k\omega\,\tau} d\tau$$

Here it is seen (: Appendix) that the integral between { } is just equal to one, provided that the interval $\left[-\frac{1}{2}L, +\frac{1}{2}L\right]$ is large enough - "approximates infinity", so to speak - when compared with σ: $\left[-\frac{1}{2}L/\sigma, +\frac{1}{2}L/\sigma\right] \approx [-\infty, +\infty]$. Then:

$$\begin{bmatrix} \overline{c}_x(k) \\ \overline{c}_y(k) \end{bmatrix} = e^{-\frac{1}{2}(\sigma k\omega)^2} \frac{1}{L} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} e^{i\,k\omega\,\tau} d\tau$$

This is the end-result:

$$\begin{bmatrix} \overline{c}_x(k) \\ \overline{c}_y(k) \end{bmatrix} = e^{-\frac{1}{2}(\sigma k\omega)^2} \begin{bmatrix} c_x(k) \\ c_y(k) \end{bmatrix} \qquad \text{where:} \quad \omega = \frac{2\pi}{L}$$

Where $[c_x, c_y]$ denote the Fourier coefficients of the "sharpened" contour. With other words: the Fourier coefficients of the fuzzyfied contour are just equal to those of the sharpened contour, provided that the latter are multiplied with a factor $exp(-\frac{1}{2}\sigma^2 k^2 \omega^2)$. Expressed somewhat differently: the function $exp(-\frac{1}{2}\sigma^2 \omega_k^2)$ acts as a *filter* for the frequencies $\omega_k = k.2\pi/L$. It is clear from this formula that the higher frequencies are damped rather quickly. To be more precise: they can be neglected if $k > L/\sigma$, because then $exp(-\frac{1}{2}\sigma^2 k^2 \omega^2)$ is smaller than $exp(-2\,\pi^2) \approx 10^{-9}$. This means that L/σ - length divided by spread - is actually an *upper bound* for the number of significant terms in the Fourier series of a fuzzyfied contour.

Lissajous Ellipses

The Fourier series of a closed contour can be written as follows:

$$x(t) = \frac{1}{2} A_x(0) + \sum_{k=1}^{\infty} x_k(t) \quad \text{and} \quad y(t) = \frac{1}{2} A_y(0) + \sum_{k=1}^{\infty} y_k(t)$$

Where:

$$x_k(t) = A_x(k)\, cos\left(k\frac{2\pi}{L}t\right) + B_x(k)\, sin\left(k\frac{2\pi}{L}t\right)$$

$$y_k(t) = A_y(k)\, cos\left(k\frac{2\pi}{L}t\right) + B_y(k)\, sin\left(k\frac{2\pi}{L}t\right)$$

But let us first recall the definition of the first order moment - also known as mean or midpoint or center of gravity:

$$\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} = \frac{1}{L} \int_0^L \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} dt$$

Here L is the length of the contour. Zero'th order Fourier coefficients are recognized at the right hand side. It follows that:

$$\frac{1}{2} \begin{bmatrix} A_x(0) \\ A_y(0) \end{bmatrix} = \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}$$

Now create a shorthand notation, with $x = x_k(t)$, $y = y_k(t)$, $a_x = A_x(k)$, $b_x = B_x(k)$ and $\omega = k\, 2\pi/L$. And concentrate on the higher order Fourier terms (with $k > 0$):

$$x = a_x\, cos(\omega t) + b_x\, sin(\omega t)$$

$$y = a_y\, cos(\omega t) + b_y\, sin(\omega t)$$

Multiply the first equation with a_y, the second with a_x and substract. The result is:

$$a_y x - a_x y = (a_y b_x - a_x b_y)\, sin(\omega t) \quad \Longrightarrow \quad sin(\omega t) = \frac{a_y x - a_x y}{a_y b_x - a_x b_y}$$

Multiply the first equation with b_y, the second with b_x and substract. The result is:

$$b_y x - b_x y = (b_y a_x - b_x a_y)\, cos(\omega t) \quad \Longrightarrow \quad cos(\omega t) = \frac{b_y x - b_x y}{b_y a_x - b_x a_y}$$

Now use the well known identity:

$$cos^2(\omega t) + sin^2(\omega t) = 1$$

Giving:

$$\left(\frac{b_y x - b_x y}{b_y a_x - b_x a_y}\right)^2 + \left(\frac{a_y x - a_x y}{a_y b_x - a_x b_y}\right)^2 = 1$$

This is a *unit circle* $(x')^2+(y')^2 = 1$ in a transformed coordinate system, which is given by:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} b_y & -b_x \\ -a_y & a_x \end{bmatrix} / (b_y a_x - b_x a_y) \begin{bmatrix} x \\ y \end{bmatrix}$$

The inverse transformation is:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_x & b_x \\ a_y & b_y \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = x' \begin{bmatrix} a_x \\ a_y \end{bmatrix} + y' \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

A unit circle which is distorded by such a linear transformation can only be an *ellipse*. It is concluded herefrom that the Lissajous series of any closed contour consists of a superposition of ellipses, each of which is traversed with a different speed. More precisely. Starting at the *midpoint* with $k = 0$, make $k := k+1$ and create an ellipse (k) around that midpoint, with skewed axes $\vec{a}(k), \vec{b}(k)$. Take the endpoint of the vector $\vec{a}.cos(k\omega t) + \vec{b}.sin(k\omega t)$ as a new midpoint and repeat the procedure, starting again at $k := k+1$.

Change of Phase

The choice of the origin of the running parameter in a contour, where it starts to run, or $t = 0$ in $(x(t), y(t))$, is quite arbitrary. It may be questioned therefore how Fourier coefficients A, B are modfied due to a *phase change*, that is: the choice of a different origin $t = \tau$. Straightforward calculation shows that it goes like this:

$$\begin{matrix} A\{f(t+\tau)\} \\ B\{f(t+\tau)\} \end{matrix} = \frac{1}{\frac{1}{2}L} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} f(t+\tau) \begin{matrix} cos(\omega t) \\ sin(\omega t) \end{matrix} \, dt =$$

$$\frac{1}{\frac{1}{2}L} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} f(t+\tau) \begin{matrix} cos(\omega\,[t+\tau-\tau]) \\ sin(\omega\,[t+\tau-\tau]) \end{matrix} \, d(t+\tau) =$$

$$\frac{1}{\frac{1}{2}L} \int_{-\frac{1}{2}L+\tau}^{+\frac{1}{2}L+\tau} f(p) \begin{matrix} cos(\omega\,[p-\tau]) \\ sin(\omega\,[p-\tau]) \end{matrix} \, dp = \frac{1}{\frac{1}{2}L} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} f(t) \begin{matrix} cos(\omega\,[t-\tau]) \\ sin(\omega\,[t-\tau]) \end{matrix} \, dt =$$

$$\frac{1}{\frac{1}{2}L} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} f(t) \begin{bmatrix} cos(\omega t)cos(\omega\tau) + sin(\omega t)sin(\omega\tau) \\ sin(\omega t)cos(\omega\tau) - cos(\omega t)sin(\omega\tau) \end{bmatrix} dt =$$

$$\begin{bmatrix} cos(\omega\tau) \\ -sin(\omega\tau) \end{bmatrix} \frac{1}{\frac{1}{2}L} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} f(t)cos(\omega t)\, dt + \begin{bmatrix} sin(\omega\tau) \\ cos(\omega\tau) \end{bmatrix} \frac{1}{\frac{1}{2}L} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} f(t)sin(\omega t)\, dt$$

$$\Longrightarrow \quad \begin{bmatrix} A\{f(t+\tau)\} \\ B\{f(t+\tau)\} \end{bmatrix} = \begin{bmatrix} cos(\omega\tau) & sin(\omega\tau) \\ -sin(\omega\tau) & cos(\omega\tau) \end{bmatrix} \begin{bmatrix} A\{f(t)\} \\ B\{f(t)\} \end{bmatrix}$$

With other words: shifting a function over a distance (τ) in real (1-D or 2-D) space corresponds with rotating its vector (A, B) of Fourier coefficients over an angle with magnitude $(\omega\tau)$. With this knowledge, write the Lissajous ellipses as follows:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} b_y & -a_y \\ -b_x & a_x \end{bmatrix} / D \begin{bmatrix} b_y & -b_x \\ -a_y & a_x \end{bmatrix} / D \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

Where $D = b_y a_x - b_x a_y$ is a determinant. It follows that:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a_x & b_x \\ a_y & b_y \end{bmatrix}^{-T} \begin{bmatrix} a_x & b_x \\ a_y & b_y \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

Where T denotes the transpose. With the rule $(AB)^{-1} = B^{-1}A^{-1}$, it follows that:

$$\begin{bmatrix} x & y \end{bmatrix} \left(\begin{bmatrix} a_x & b_x \\ a_y & b_y \end{bmatrix} \begin{bmatrix} a_x & b_x \\ a_y & b_y \end{bmatrix}^T \right)^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

Now replace the coefficients (a, b) with coefficients which are subject to a change of phase. That is: the running parameter t is shifted over a distance τ along the curve. This means that the accompanying Fourier coefficients are rotated over an angle $(\omega\tau)$:

$$\begin{bmatrix} A\{x(t+\tau)\} \\ B\{x(t+\tau)\} \end{bmatrix} = \begin{bmatrix} cos(\omega\tau) & sin(\omega\tau) \\ -sin(\omega\tau) & cos(\omega\tau) \end{bmatrix} \begin{bmatrix} A\{x(t)\} \\ B\{x(t)\} \end{bmatrix}$$

And:

$$\begin{bmatrix} A\{y(t+\tau)\} \\ B\{y(t+\tau)\} \end{bmatrix} = \begin{bmatrix} cos(\omega\tau) & sin(\omega\tau) \\ -sin(\omega\tau) & cos(\omega\tau) \end{bmatrix} \begin{bmatrix} A\{y(t)\} \\ B\{y(t)\} \end{bmatrix}$$

This can be summarized as:

$$\begin{bmatrix} a_x(+\tau) & a_y(+\tau) \\ b_x(+\tau) & b_y(+\tau) \end{bmatrix} = \begin{bmatrix} cos(\omega\tau) & sin(\omega\tau) \\ -sin(\omega\tau) & cos(\omega\tau) \end{bmatrix} \begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix}$$

Repeat the expression for the ellipse, after a slight modification:

$$\begin{bmatrix} x & y \end{bmatrix} \left(\begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix}^T \begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix} \right)^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

Shift the running parameter over a distance (τ) and apply the matrix transposition rule $(AB)^T = B^T A^T$, then:

$$\begin{bmatrix} a_x(+\tau) & a_y(+\tau) \\ b_x(+\tau) & b_y(+\tau) \end{bmatrix}^T \begin{bmatrix} a_x(+\tau) & a_y(+\tau) \\ b_x(+\tau) & b_y(+\tau) \end{bmatrix} =$$

$$\begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix}^T \begin{bmatrix} cos(\omega\tau) & sin(\omega\tau) \\ -sin(\omega\tau) & cos(\omega\tau) \end{bmatrix}^T \begin{bmatrix} cos(\omega\tau) & sin(\omega\tau) \\ -sin(\omega\tau) & cos(\omega\tau) \end{bmatrix} \begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix} =$$

$$\begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix}^T \begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix}$$

Because the transpose of an orthogonal matrix is equal to its inverse and the product of the original matrix and its inverse is equal to the unity matrix. This proves that the orientation as well as the shape of any Lissajous ellipse is *invariant* for a shift in the running parameter.

Ellipses Invariants

We will take a closer look now at each of the Lissajous ellipses. First recall:

$$\left(\frac{b_y x - b_x y}{b_y a_x - b_x a_y}\right)^2 + \left(\frac{a_y x - a_x y}{a_y b_x - a_x b_y}\right)^2 = 1 \quad \Longrightarrow$$

$$\frac{b_y^2 x^2 - 2 b_y b_x x y + b_x^2 y^2 + a_y^2 x^2 - 2 a_y a_x x y + a_x^2 y^2}{(a_x b_y - a_y b_x)^2} =$$

$$\frac{(b_y^2 + a_y^2) x^2 - 2(b_y b_x + a_y a_x) x y + (b_x^2 + a_x^2) y^2}{(a_x b_y - a_y b_x)^2} = 1$$

The matrix accompanying this quadratic form is:

$$Q = \begin{bmatrix} (b_y^2 + a_y^2) & -(b_y b_x + a_y a_x) \\ -(b_y b_x + a_y a_x) & (b_x^2 + a_x^2) \end{bmatrix} / (a_x b_y - a_y b_x)^2$$

The determinant of the matrix, without the denominator, is calculated:

$$(b_y^2 + a_y^2)(b_x^2 + a_x^2) - (b_y b_x + a_y a_x)^2 =$$

$$(b_y^2 b_x^2 + b_y^2 a_x^2 + a_y^2 b_x^2 + a_y^2 a_x^2) - (b_y^2 b_x^2 + 2 b_y b_x a_y a_x + a_y^2 a_x^2) =$$

$$a_x^2 b_y^2 - 2 a_x b_y a_y b_x + a_y^2 b_x^2 = (a_x b_y - a_y b_x)^2$$

The outcome just happens to be equal to the denominator. This makes inverting the matrix an easy thing to do:

$$Q^{-1} = \begin{bmatrix} (a_x^2 + b_x^2) & (b_y b_x + a_y a_x) \\ (b_y b_x + a_y a_x) & (a_y^2 + b_y^2) \end{bmatrix}$$

The axes of the ellipses have lengths which are equal to the squares of the eigenvalues of the above inverse matrix, according to the template:

$$\left(\frac{x}{\sqrt{\lambda_1}}\right)^2 + \left(\frac{y}{\sqrt{\lambda_2}}\right)^2 = 1$$

The characteristic equation is:

$$\left\| \begin{matrix} (a_x^2 + b_x^2) - \lambda & (b_y b_x + a_y a_x) \\ (b_y b_x + a_y a_x) & (a_y^2 + b_y^2) - \lambda \end{matrix} \right\| = 0 \quad \Longrightarrow \quad \lambda^2 - Sp\,\lambda + Det = 0$$

Where the *trace* of the matrix is defined by:

$$Sp = (a_x^2 + b_x^2) + (a_y^2 + b_y^2) = (a_x^2 + a_y^2) + (b_x^2 + b_y^2)$$

And its *determinant* has already been calculated to be:

$$Det = (a_x^2 + b_x^2)(a_y^2 + b_y^2) - (b_y b_x + a_y a_x)^2 = (a_x b_y - a_y b_x)^2$$

The eigenvalues are solutions of the quadratic equation, giving:

$$\lambda = \frac{Sp}{2} \pm \sqrt{\left(\frac{Sp}{2}\right)^2 - Det}$$

Since the axes of the ellipses are the square roots of these eigenvalues, it remains to show that the expression under the square root is positive (though the latter should already be clear from the fact that the characteristic matrix is symmetric and positive definite). We have implicitly found that:

$$(a_x^2 + b_x^2)(a_y^2 + b_y^2) \geq (a_x b_y - a_y b_x)^2$$

It is legal to replace, in the discriminant, the determinant by something which is bigger. And to prove that the outcome is still positive:

$$\left(\frac{Sp}{2}\right)^2 - Det \geq \left(\frac{(a_x^2 + a_y^2) + (b_x^2 + b_y^2)}{2}\right)^2 - (a_x^2 + b_x^2)(a_y^2 + b_y^2) \geq 0$$

Substitute for simplicity the abbreviations $A = a_x^2 + a_y^2$ and $B = b_x^2 + b_y^2$. Then we have to prove:

$$\left(\frac{A+B}{2}\right)^2 - AB = \frac{A^2}{4} + \frac{AB}{2} + \frac{B^2}{4} - AB = \frac{A^2}{4} - \frac{AB}{2} + \frac{B^2}{4} = \left(\frac{A-B}{2}\right)^2 \geq 0$$

Quod Erat Demonstrandum. Hence the lengths of the axes of the ellipses can safely be calculated to be:

$$\sqrt{\lambda} = \sqrt{\frac{Sp}{2} \pm \sqrt{\left(\frac{Sp}{2}\right)^2 - Det}}$$

With:

$$Sp = (a_x^2 + a_y^2) + (b_x^2 + b_y^2) \quad \text{and} \quad Det = (a_x b_y - a_y b_x)^2$$

As far as the accompanying eigenvectors are concerned, it makes no difference, essentially, whether the first or the second row of the matrix is used:

$$(a_x^2 + b_x^2 - \lambda)x + (b_y b_x + a_y a_x)y = 0 \quad \text{or} \quad (b_y b_x + a_y a_x)x + (a_y^2 + b_y^2 - \lambda)y = 0$$

If the first one is taken, then:

$$\left[\frac{1}{2}(a_x^2 + b_x^2) - \frac{1}{2}(a_y^2 + b_y^2) \pm \sqrt{D}\right] x + (b_y b_x + a_y a_x)y = 0$$

Where $D = (Sp/2)^2 - Det$. Define even more shorthand variables:

$$A = \frac{1}{2}(a_x^2 + b_x^2) - \frac{1}{2}(a_y^2 + b_y^2) \pm \sqrt{D} \quad \text{and} \quad B = (b_y b_x + a_y a_x)$$

And form normed (length = 1) solutions. Then the eigenvectors are:

$$Ax + By = 0 \quad \Longrightarrow \quad (x, y) = \frac{(B, -A)}{\sqrt{A^2 + B^2}}$$

If a and b are the (lengths of the mutually perpendicular main) axes of an ellipse, then its area is given by $\pi.a.b$. In our case $a = \sqrt{\lambda_1}$ and $b = \sqrt{\lambda_2}$, thus $\pi.a.b = \pi\sqrt{\lambda_1.\lambda_2} = \pi.\sqrt{Det} = \pm\,(b_x a_y - a_x b_y)$. A more precise analysis will remove the $\pm$ sign ambiguity, in the next subsection.
The two quantities $\sqrt{a_x^2 + a_y^2 + b_x^2 + b_y^2}$ and $(b_x a_y - a_x b_y)$ may be called the *Lissajous Invariants* of (the k-th term of) the Fourier expansion. They are quite independent of the running parameter (t) and the origin where it starts to run at the curve, *phase independent* so to speak. They are also independent of any translations & rotations of the coordinate system.

Bessel's Inequality

The following facts are well known from the theory of Fourier Series. Bessel's Inequality reads as follows:

$$\frac{1}{\frac{1}{2}L} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} f^2(t)\,dt \;\geq\; \frac{1}{2}a_0^2 + \sum_{k=1}^{\infty} \left(a_k^2 + b_k^2\right) \;>\; \frac{1}{2}a_0^2 + \sum_{k=1}^{N} \left(a_k^2 + b_k^2\right)$$

Here a and b are the Fourier Coefficients of the function f. The equal sign is valid for piecewise continuous functions which are differentiable to the left and to the right (except for a few isolated points) and the inequality is then called the Theorem of Parceval. Parceval's Theorem may be conceived as the equivalent of the famous theorem by Pythagoras, for infinite dimensional vector spaces. Kind of an integral difference between the function and its Fourier series is given by:

$$\frac{1}{\frac{1}{2}L} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} \left\{ f(t) - \left[\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k cos(k\omega t) + b_k sin(k\omega t) \right] \right\}^2 dt =$$

$$\frac{1}{\frac{1}{2}L} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} f^2(t)\,dt - \left[\frac{1}{2}a_0 + \sum_{k=1}^{N} \left(a_k^2 + b_k^2\right) \right] > 0$$

Leading to a suitable criterion for the accuracy of the Fourier approximation, which is the (square root of the) above divided by:

$$\frac{1}{\frac{1}{2}L} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} f^2(t)\,dt$$

For the Lissajous approximation of piecewise continuous curves, we must replace $f(t)$ by $x(t)$ as well as $y(t)$, and sum up the results.

An interesting side effect is that Parceval's theorem will enable us to express the second order moments of a closed curve in its Fourier Coefficients only:

$$\left[\begin{array}{c} \overline{x^2} \\ \overline{y^2} \end{array}\right] = \frac{1}{L}\int_0^L \left[\begin{array}{c} x^2(t) \\ y^2(t) \end{array}\right] dt$$

The integrals differ from those in the Bessel/Parceval Theorem only by a factor $1/2$. Thus we find:

$$\overline{x^2} = \frac{1}{4}A_x^2(0) + \frac{1}{2}\sum_{k=1}^{\infty}\left[A_x^2(k) + B_x^2(k)\right]$$

Where $A_x^2(0)/4$ is recognized as a second order moment of the midpoint. If we restrict attention to moments of intertia *with respect to* the center of gravity, as is common practice, then:

$$\overline{x^2} - \overline{x}^2 = \frac{1}{2}\sum_{k=1}^{\infty}\left[A_x^2(k) + B_x^2(k)\right]$$

Quite analagously:

$$\overline{y^2} - \overline{y}^2 = \frac{1}{2}\sum_{k=1}^{\infty}\left[A_y^2(k) + B_y^2(k)\right]$$

And since $\overline{xy}$ is like an inner product in an infinite dimensional vector space with an orthogonal base of *sin* and *cos* functions, we may also safely infer that:

$$\overline{xy} - \overline{x}\,\overline{y} = \frac{1}{2}\sum_{k=1}^{\infty}\left[A_x(k)A_y(k) + B_x(k)B_y(k)\right]$$

The *area* A inside a contour may be defined analytically as:

$$A = \int_0^L (x.dy - y.dx) = \int_0^L (x.y' - x'.y)dt$$

Here the functions $x(t)$ and $y(t)$ will developed again as Fourier series. The integrals then behave as inner products with an orthogonal base:

$$A = \frac{1}{2}L(\vec{x}\cdot\vec{y'}) - \frac{1}{2}L(\vec{x'}\cdot\vec{y})$$

Resulting in expressions like with $\overline{xy}$:

$$A = \frac{1}{2}L\,.\,\frac{1}{2}\sum_{k=1}^{\infty}\left[A_x(k)A'_y(k) + B_x(k)B'_y(k)\right]$$

$$-\frac{1}{2}L\,.\,\frac{1}{2}\sum_{k=1}^{\infty}\left[A'_x(k)A_y(k) + B'_x(k)B_y(k)\right]$$

Body workout:

$$A = \frac{1}{4}L \sum_{k=1}^{\infty} [A_x(k)(-k\omega)B_y(k) + B_x(k)(k\omega)A_y(k)$$

$$-(-k\omega)B_x(k)A_y(k) - (k\omega)A_x(k)B_y(k)] =$$

$$\frac{1}{4}L \; \frac{2\pi}{L} \sum_{k=1}^{\infty} k\,[-A_x(k)B_y(k) + B_x(k)A_y(k) + B_x(k)A_y(k) - A_x(k)B_y(k)] =$$

$$\pi \sum_{k=1}^{\infty} k\,[B_x(k)A_y(k) - A_x(k)B_y(k)] = \sum_{k=1}^{\infty} k.\pi Det_k$$

Here πDet_k is recognized as the area of a Lissajous Ellipse. The variable k may be interpreted as the number of times the circumference of a such an ellipse is traversed, while the original curve is traversed exactly one time. Hence k may also be called the *winding number* of the Lissajous Ellipse. This leads to the remarkably simple result that the area inside a contour is equal to the sum of the areas of the accompanying Lissajous Ellipses, where each area must be multiplied by its proper winding number.

Complex Formulation

The definition of a Lissajous series is repeated:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A_x(0) \\ A_y(0) \end{bmatrix} + \sum_{k=1}^{\infty} \left\{ \begin{bmatrix} A_x(k) \\ A_y(k) \end{bmatrix} cos\left(k\frac{2\pi}{L}t\right) + \begin{bmatrix} B_x(k) \\ B_y(k) \end{bmatrix} sin\left(k\frac{2\pi}{L}t\right) \right\}$$

Where the *Fourier coefficients* are to be evaluated, according to:

$$\begin{bmatrix} A_x(k) \\ A_y(k) \end{bmatrix} = \frac{1}{\frac{1}{2}L}\int_0^L \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} cos\left(k\frac{2\pi}{L}t\right)\,dt$$

$$\begin{bmatrix} B_x(k) \\ B_y(k) \end{bmatrix} = \frac{1}{\frac{1}{2}L}\int_0^L \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} sin\left(k\frac{2\pi}{L}t\right)\,dt$$

Complex Formulation starts here. It follows that:

$$x(t) + i.y(t) = \frac{1}{2}\,[A_x(0) + i.A_y(0)]$$

$$+\sum_{k=1}^{\infty} \left\{ [A_x(k) + i.A_y(k)]\,cos\left(k\frac{2\pi}{L}t\right) + [B_x(k) + i.B_y(k)]\,sin\left(k\frac{2\pi}{L}t\right) \right\}$$

Define the angular (ground) frequency $\omega = 2\pi/L$ and complex quantities

$$\gamma(t) = x(t) + i.y(t)$$

$$A(k) = A_x(k) + i.A_y(k) \quad ; \quad B(k) = B_x(k) + i.B_y(k)$$

Then the above is the same as:

$$\gamma(t) = \frac{1}{2}A(0) + \sum_{k=1}^{\infty} \{A(k)\, cos(k\omega t) + B(k)\, sin(k\omega t)\}$$

Substitute well known complex equivalents for sine and cosine:

$$\gamma(t) = \frac{1}{2}A(0) + \sum_{k=1}^{\infty} \left\{ A(k) \frac{e^{+ik\omega t} + e^{-ik\omega t}}{2} + B(k)\, \frac{e^{+ik\omega t} - e^{-ik\omega t}}{2i} \right\} \quad \Longrightarrow$$

$$\gamma(t) = \frac{1}{2}A(0) + \frac{1}{2}\sum_{k=1}^{\infty} \left\{ [A(k) - i.B(k)]\, e^{+ik\omega t} + [A(k) + i.B(k)]\, e^{-ik\omega t} \right\}$$

From the definitions of the Fourier coefficients and those of A and B :

$$A(k) = \frac{1}{\frac{1}{2}L} \int_0^L \gamma(t)\, cos(k\omega t)\, dt \quad ; \quad B(k) = \frac{1}{\frac{1}{2}L} \int_0^L \gamma(t)\, sin(k\omega t)\, dt$$

Define:

$$c_k = \frac{A(k) - i.B(k)}{2} = \frac{1}{L} \int_0^L \gamma(t)\; [cos(k\omega t) - i.sin(k\omega t)]\, dt \quad \Longrightarrow$$

$$c_k = \frac{1}{L} \int_0^L \gamma(t)\, e^{-ik\omega t} dt$$

Consequently, for values of k negative and zero:

$$c_{-k} = \frac{1}{L} \int_0^L \gamma(t)\, e^{+ik\omega t} dt =$$

$$\frac{1}{L} \int_0^L \gamma(t)\; [cos(k\omega t) + i.sin(k\omega t)]\, dt = \frac{A(k) + i.B(k)}{2}$$

$$c_0 = \frac{1}{L} \int_0^L \gamma(t) dt = \frac{1}{2}A(0)$$

Giving as the end result:

$$\gamma(t) = \sum_{k=-\infty}^{+\infty} c_k\, e^{ik\omega t}$$

Where as before:

$$c_k = \frac{1}{L} \int_0^L \gamma(t)\, e^{-ik\omega t} dt$$

Great simplification, but we are not finished, not yet. Instead of t, take τ as the running parameter, where $\tau = 2\pi.t/L = \omega\, t$. Then:

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} \gamma(\tau)\, e^{-ik\tau}\, d\tau$$

$$\gamma(\tau) = \sum_{k=-\infty}^{+\infty} c_k\, e^{ik\tau} = \sum_{k=-\infty}^{+\infty} c_k\, z^k \qquad \text{where} \quad z = e^{ik\tau}$$

Define the complex function f as a Laurent series:

$$f(z) = \sum_{k=-\infty}^{+\infty} c_k\, z^k \quad \Longrightarrow \quad \gamma(t) = f(e^{it})$$

And the coefficients c_k are evaluated as:

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\left(e^{it}\right)^{-k} dt$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(e^{it})}{\left(e^{it}\right)^{k+1}} d(e^{it}) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{k+1}} dz$$

Where $\gamma(t) = e^{it}$. In "Complex Made Simple" by David C. Ullrich, it is read on page 42: "the restriction of a power series [Laurent series actually (HdB)] to a circle is a Fourier series". And the last formula above is a special case of Cauchy's Integral Formula (e.g. Theorem 2.6 on CMS page 28).

Appendix

The fact that the integral between curly brackets is equal to one is a simple, but nevertheless remarkable, application of Cauchy's Integral Theorem. It is conjectured that, for L large enough:

$$\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}\left[t-(\tau+\sigma^2 i\, k\omega)\right]^2/\sigma^2} dt = 1$$

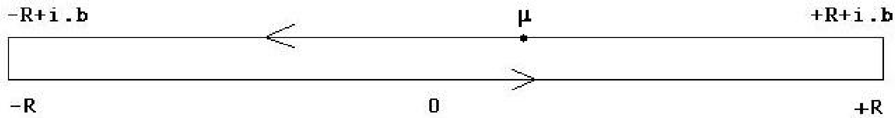
With real spread $\sigma > 0$. The integral is written for our purpose as:

$$\int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}(t-\mu)^2/\sigma^2} dt = 1$$

The conjecture is well known to be true for μ being a real number. But, in our case, μ is complex instead of real. Let's write it as $\mu = a + i.b$. And consider the following line integral in the complex plane. It is zero because of Cauchy's Integral Theorem for functions holomorphic in the whole complex plane:

$$\int_\gamma \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}(z-\mu)^2/\sigma^2} dz = 0$$

Where the path γ is defined as follows, with $-R < \Re(\mu) < +R$:



And thus the path integral consists of the following parts:

$$\int_{\gamma} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(z-\mu)^2/\sigma^2} dz =$$

$$\int_{\gamma_1} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(z-\mu)^2/\sigma^2} dz + \int_{\gamma_2} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(z-\mu)^2/\sigma^2} dz +$$

$$\int_{\gamma_3} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(z-\mu)^2/\sigma^2} dz + \int_{\gamma_4} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(z-\mu)^2/\sigma^2} dz$$

Where:

$$\int_{\gamma_1} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(z-\mu)^2/\sigma^2} dz = \int_{-R}^{+R} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}[x-(a+i.b)]^2/\sigma^2} dx$$

$$\int_{\gamma_2} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(z-\mu)^2/\sigma^2} dz = \int_{0}^{b} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}[+R+i.y-(a+i.b)]^2/\sigma^2} i.dy$$

$$\int_{\gamma_3} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(z-\mu)^2/\sigma^2} dz = \int_{+R}^{-R} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}[x-a]^2/\sigma^2} dx$$

$$\int_{\gamma_4} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(z-\mu)^2/\sigma^2} dz = \int_{b}^{0} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}[-R+i.y-(a+i.b)]^2/\sigma^2} i.dy$$

The integral along γ_3 is the well known one and is equal to 1. If the integrals along γ_2 and γ_4 can be "talked zero" (i.e. argued to converge to zero if R approaches infinity) then we are finished, because in that case:

$$\int_{-R}^{+R} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}[x-(a+i.b)]^2/\sigma^2} dx = \int_{-R}^{+R} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}[x-a]^2/\sigma^2} dx = 1$$

So let's finish the job. The ML theorem is employed for this purpose.

$$\left| \int_{\gamma_{2,4}} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(z-\mu)^2/\sigma^2} dz \right| = \left| \pm \int_{0}^{b} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}[\pm R+i.y-(a+i.b)]^2/\sigma^2} i.dy \right| \le$$

$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left[(R-a)^2+b^2\right]^2/\sigma^2} .b \quad \to 0 \qquad \text{for} \quad R \to \infty$$

After having done all this comes Robert Israel, who simply says the following:

```
Assuming sigma > 0, it's true. By easy estimates, the integral is analytic
(as a function of mu). An entire function that is constant on the real axis
is constant.
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http://groups.google.nl/group/sci.math/browse_frm/thread/d4226628f043be82/
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