Airy R. Bean's Problem

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From: "Airy R. Bean" <me@privacy.net>
Newsgroups: comp.dsp,alt.engineering.electrical,sci.math,uk.radio.amateur
Subject: Re: Disappointed
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Message-ID: <31r0vfF3f7215U4@individual.net>

Well, that's what I suggested.

It then leaves us, because it is now continuous,
with the need to find an identity for
the anti-derivative of a(t).b(t).c(t). I know of none
apart from applying Integration By Parts twice, but, as has
been pointed out, it is erroneous to evaluate a definite integral
prematurely before determining the total anti-derivative.

Those (Ullrich, Daestrom) who tried to correct me upon the
Integration By Parts themselves made the error of arguing
a posteriori by applying the result of a definite integral in order
to determine an anti-derivative upon which the definite
integral depended.

"Han de Bruijn" <Han.deBruijn@DTO.TUDelft.NL> wrote in message
news:cp98nt$ksg$1@news.tudelft.nl...
> Airy R. Bean wrote:
>
> > We are stuck on the evaluation of int -oo^+oo f(t).d(t).e^(-st)
>
> No guarantee, just a try. Maybe the misunderstanding is in the d(t).
> As you define it, is it the Gaussian function with a very small spread?
>
> Han de Bruijn
>
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OK. Let's do just that. Define the delta function - what's in a name - by:

$$\delta(t) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}t^2/\sigma^2}$$

The laplace integral to be calculated is then:

$$\int_{-\infty}^{+\infty} f(t)\,.\,\delta(t-T)\,.\,e^{-s.t}\,dt = \int_{-\infty}^{+\infty} f(t)\,.\,\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(t-T)^2/\sigma^2}\,.\,e^{-s.t}\,dt$$

$$= \int_{-\infty}^{+\infty} f(t) \, . \, \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}(t-T)^2/\sigma^2 - s.t} \; dt$$

The exponent of the exponential function is worked out separately, as follows:

$$-\frac{1}{2}(t-T)^2/\sigma^2 - s.t =$$

$$-\frac{1}{2}\left(\frac{t-T}{\sigma}\right)^2 - s.\sigma\left(\frac{t-T}{\sigma}\right) - \frac{1}{2}\,(s.\sigma)^2 - s.T + \frac{1}{2}\,(s.\sigma)^2$$

$$= -\frac{1}{2}\left[\left(\frac{t-T}{\sigma}\right) + s.\sigma\right]^2 \; - s.T + \frac{1}{2}(s.\sigma)^2$$

$$= -\frac{1}{2}\left[t - (T - s.\sigma^2)\right]^2 / \sigma^2 \; - s.T + \frac{1}{2}(s.\sigma)^2$$

The term $(T - s.\sigma^2)$ will be abbreviated as μ . Then it follows that:

$$\frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}(t-T)^2/\sigma^2 - s.t} = \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}(t-\mu)^2/\sigma^2} \; . \; e^{-s.T + \frac{1}{2}(s.\sigma)^2}$$

Giving for the integral:

$$\int_{-\infty}^{+\infty} f(t) \, . \, \delta(t-T) \, . \, e^{-s.t} \; dt =$$

$$\int_{-\infty}^{+\infty} f(t) \, . \, \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}(t-\mu)^2/\sigma^2} \; dt \quad . \quad e^{-s.T + \frac{1}{2}(s.\sigma)^2}$$

The laplace integral has been transformed into an integral over the product of $f(t)$ with a normal density distribution. The latter has an expectation value $\mu = T - s.\sigma^2$. And a spread σ. The next step is to develop $f(t)$ in a Taylor series around the mean μ of the normal distribution. Only the first three terms are assumed significant, because we will let $\sigma \rightarrow 0$ later on.

$$f(t) \approx f(\mu) + (t-\mu).f'(\mu) + \frac{1}{2}(t-\mu)^2.f''(\mu)$$

Substitute this into the latter integral, then:

$$\int_{-\infty}^{+\infty} f(t) \, . \, \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}(t-\mu)^2/\sigma^2} \; dt \approx$$

$$f(\mu) \; . \; \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}(t-\mu)^2/\sigma^2} \; d(t-\mu)$$

$$+ \, f'(\mu) \; . \; \int_{-\infty}^{+\infty} (t-\mu) \; \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}(t-\mu)^2/\sigma^2} \; d(t-\mu)$$

$$+ \frac{1}{2} f''(\mu) \, . \int_{-\infty}^{+\infty} (t-\mu)^2 \, \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}(t-\mu)^2/\sigma^2} \; d(t-\mu)$$

The integrals are recognized as moments of the normal density distribution (which are assumed to be well known). Therefore the right hand side becomes:

$$f(\mu) \, . \, 1 + f'(\mu) \, . \, 0 + \frac{1}{2} f''(\mu) \, . \, \sigma^2 = f(\mu) + \frac{1}{2} f''(\mu) \, . \, \sigma^2$$

Substitute this in the expression that we found for the Laplace integral:

$$\int_{-\infty}^{+\infty} f(t) \, . \, \delta(t-T) \, . \, e^{-s.t} \; dt =$$

$$\left[\int_{-\infty}^{+\infty} f(t) \, . \, \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}(t-\mu)^2/\sigma^2} \; dt \right] e^{-s.T + \frac{1}{2}(s.\sigma)^2}$$

$$\approx \left[f(\mu) + \frac{1}{2} f''(\mu) \, . \, \sigma^2 \right] e^{-s.T + \frac{1}{2}(s.\sigma)^2}$$

We are almost there. Remember that $\mu = T - s.\sigma^2$. The only thing we have to do now is take the limit for $\sigma \to 0$. This reduces μ to T and therefore $f(\mu)$ to $f(T)$. It is concluded that:

$$\int_{-\infty}^{+\infty} f(t) \, . \, \delta(t-T) \, . \, e^{-s.t} \; dt = \left[f(T) + \frac{1}{2} f''(T) \, . \, 0^2 \right] e^{-s.T + \frac{1}{2}(s.0)^2} \qquad \Longrightarrow$$

$$\int_{-\infty}^{+\infty} f(t) \, . \, \delta(t-T) \, . \, e^{-s.t} \; dt = f(T) \, . \, e^{-s.T}$$

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch.