Fuzzy Geometries

The *VaagZien* demonstration program was developed, because I have been fascinated quite some time by the notion of *Fuzzy Optics*. The subject already fills a chapter in my 1995 (Dutch) book:

`http://huizen.dto.tudelft.nl/deBruijn/chap6/fuzzy.htm`

Further thinking has given rise to a couple of new delightful insights, which should not be missing in my Sections on Pure Applicable Mathematics (SPAM ;-) . This report covers some of the theory that is required before any programming language can be set to work.

Two-dimensional moments

Consider an arbitrary 2-D distribution of N points (x_k, y_k) in the plane, also called a collection of points, or points cloud. A quantity called weight or mass m_k is associated with each of these points. The total weight or mass M of the points is given by the sum of the weights:

$$M = \sum_{k=1}^{N} m_k = \sum_k m_k$$

It will be assumed in the sequel that the weights are always positive, meaning that they can be *normed*. Such normed weights w_k are defined by:

$$w_k = \frac{m_k}{M} \quad \Longrightarrow \quad 0 \le w_k \le 1 \quad \text{and} \quad \sum_k w_k = 1$$

We can define a spot, called the midpoint, center of mass, or whatever name is to be preferred, as follows:

$$\mu_x = \overline{x} = \sum_k w_k x_k \quad \text{and} \quad \mu_y = \overline{y} = \sum_k w_k y_k$$

In addition to the above discrete quantities, there also exist *continuum versions* of them. The only difference being that the latter are defined by (definite) integrals instead of sums:

$$\iint w(x,y)\,dx\,dy = 1 \quad \Longrightarrow$$

$$\mu_x = \overline{x} = \iint w(x,y)x\,dx\,dy \quad \text{and} \quad \mu_y = \overline{y} = \iint w(x,y)y\,dx\,dy$$

It is clear from the outset, however, that such integrals are just limiting cases of discrete sums. Hence subsequent results for the discrete case will also be valid

for the continuous case.
The midpoint takes a special position inside (: not necessarily) the points cloud, as it is the weighted mean value of all positions of the points in the cloud. The midpoint is also called a *first order moment* of the points cloud. It's easy to conceive the weighted mean value of other quantities, however. Most useful are the so-called *second order moments*, also known as the *moments of inertia*, due to their application in Classical Mechanics. First, mean values of x^2 , y^2 and $x.y$ are defined as follows:

$$\overline{x^2} = \sum_k w_k x_k^2 \quad \text{or} \quad \overline{x^2} = \iint w(x,y) x^2 \, dx \, dy$$

$$\overline{y^2} = \sum_k w_k y_k^2 \quad \text{or} \quad \overline{y^2} = \iint w(x,y) y^2 \, dx \, dy$$

$$\overline{x.y} = \sum_k w_k x_k . y_k \quad \text{or} \quad \overline{x.y} = \iint w(x,y) x.y \, dx \, dy$$

The second order momenta are defined with respect to an origin (p,q):

$$\sigma_{xx}(p) = \sum_k w_k (x_k - p)^2 \quad \text{and} \quad \sigma_{yy}(q) = \sum_k w_k (y_k - q)^2$$

$$\sigma_{xy}(p,q) = \sum_k w_k (x_k - p).(y_k - q)$$

The continuous counterparts are:

$$\sigma_{xx}(p) = \iint w(x,y)(x-p)^2 \, dx \, dy \quad \text{and} \quad \sigma_{yy}(q) = \iint w(x,y)(y-q)^2 \, dx \, dy$$

$$\sigma_{xy}(p,q) = \iint w(x,y)(x-p).(y-q)^2 \, dx \, dy$$

For the moment being, attention will be restricted to the second order moment for the x-direction only: σ_{xx} . And it will be shown that, for this quantity, there exists a preferrable origin, which turns out to be the midpoint of the points distribution. Here goes:

$$\sigma_{xx}(p) = \sum_k w_k (x_k - p)^2 = \sum_k w_k x_k^2 - 2p \sum_k w_k x_k + p^2 =$$

$$\overline{x^2} - 2p\overline{x} + p^2 = \left[\overline{x^2} - \overline{x}^2\right] + \left[\overline{x}^2 - 2p\overline{x} + p^2\right]$$

The first term between square brackets [] can be worked out as follows:

$$\left[\sum_k w_k x_k^2 - \left(\sum_k w_k x_k \right)^2 \right] =$$

$$\sum_k w_k x_k^2 - 2\sum_k w_k \left(\sum_k w_k x_k\right) x_k + \sum_k w_k \left(\sum_k w_k x_k\right)^2 =$$

$$\sum_k w_k \left[x_k^2 - 2\left(\sum_k w_k x_k\right) x_k + \left(\sum_k w_k x_k\right)^2\right] =$$

$$\sum_k w_k \left[x_k - \left(\sum_k w_k x_k\right)\right]^2 = \sum_k w_k \left(x_k - \overline{x}\right)^2$$

And the second term between square brackets [ ] can be worked out as follows:

$$\left[\overline{x}^2 - 2p\overline{x} + p^2\right] = \left(\overline{x} - p\right)^2$$

Conclusion:

$$\sum_k w_k (x_k - p)^2 = \sum_k w_k \left(x_k - \overline{x}\right)^2 + \left(\overline{x} - p\right)^2$$

Then we see that the first term is positive, because it is a sum of squares. But also the second term is a square and hence positive. The latter assumes a minimum if it is exactly zero, that is: $p = \overline{x}$. Formally:

$$\sum_k w_k (x_k - p)^2 = \text{minimum}(p) \quad \Longleftrightarrow \quad p = \overline{x} = \sum_k w_k x_k$$

The physical interpretation of the above is that a moment of inertia assumes a minimal value with respect to the origin if that origin is coincident with the center of mass. A moment of intertia with respect to an origin which is different from the center of mass can be expressed as the sum of two moments: one which expresses the moment of intertia with respect to the midpoint plus one which expresses the moment of interia of the midpoint with respect to the origin. Unless explicitly stated otherwise, it will be assumed in the sequel that all moments of intertia are defined with respect to the midpoint μ_x. Then we can drop the dependence on (p) in:

$$\sigma_{xx} = \sum_k m_k (x_k - \mu_x)^2$$

Very much the same reasoning can be accomplished for the second order moment in the y-direction:

$$\sum_k w_k (y_k - q)^2 = \text{minimum}(q) \quad \Longleftrightarrow \quad q = \overline{y} = \sum_k w_k y_k$$

How about the "mixed" second order moment σ_{xy}?

$$\sigma_{xy}(\overline{x}, \overline{y}) = \sum_k w_k (x_k - \overline{x})(y_k - \overline{y}) = \sum_k w_k x_k y_k - \sum_k x_k \overline{y} - \sum_k y_k \overline{x} + \overline{x}\,\overline{y} =$$

$$\sum_k x_k y_k - \overline{x}\,\overline{y} - \overline{y}\,\overline{x} + \overline{x}\,\overline{y} \quad \Longrightarrow \quad \sigma_{xy} = \overline{xy} - \overline{x}\,\overline{y}$$

Since it will be assumed in the sequel that all moments of inertia are with respect to the midpoint (μ_x, μ_y), we can drop (p, q) in:

$$\sigma_{xx} = \sum_k w_k (x_k - \mu_x)^2 \quad \text{and} \quad \sigma_{yy} = \sum_k w_k (y_k - \mu_y)^2$$

$$\sigma_{xy} = \sum_k w_k (x_k - \mu_x)(y_k - \mu_y)$$

So far, it is less clear what kind of physical meaning should be attached to the quantity σ_{xy}, which is sometimes known as a "cross correlation". Suppose however, that we don't like it at all and that we only want to get rid of this term. How then could we accomplish such a thing ? It can certainly not be done by translation, since the origin of our coordinate system has become fixed at the midpoint. But there is another possibility. It could be done by *rotating* the coordinate system in such a way that σ'_{xy} becomes zero in the new ('primed') coordinate system. Let's give it a try. Start with:

$$\begin{cases} x' = cos(\theta)x + sin(\theta)y \\ y' = -sin(\theta)x + cos(\theta)y \end{cases}$$

Then:

$$\sigma'_{xy} = 0 \quad \Longleftrightarrow \quad \sum_k w_k x'_k y'_k =$$

$$\sum_k w_k \left[cos(\theta)x_k + sin(\theta)y_k\right]\left[-sin(\theta)x_k + cos(\theta)y_k\right] =$$

$$-cos(\theta)sin(\theta) \sum_k w_k x_k^2 + sin(\theta)cos(\theta) \sum_k w_k y_k^2$$

$$+ \left[cos^2(\theta) - sin^2(\theta)\right] \sum_k w_k x_k y_k =$$

$$-\frac{1}{2} sin(2\theta)(\sigma_{xx} - \sigma_{yy}) + cos(2\theta)\sigma_{xy} = 0$$

Resulting in:

$$tan(2\theta) = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} \qquad \text{for} \quad \sigma_{xx} \neq \sigma_{yy}$$

The other two moments of intertia, σ'_{xx} and σ'_{yy}, are then expressed into the angle θ as follows:

$$\sigma'_{xx} = \sum_k w_k (x'_k)^2 = \sum_k w_k \left[cos(\theta)x_k + sin(\theta)y_k\right]^2$$

$$= cos^2(\theta) \sum_k w_k x_k^2 + sin^2(\theta) \sum_k w_k y_k^2 + 2 sin(\theta) cos(\theta) \sum_k w_k x_k y_k$$

$$\Longrightarrow \quad \sigma'_{xx} = cos^2(\theta)\sigma_{xx} + sin^2(\theta)\sigma_{yy} + 2sin(\theta)cos(\theta)\sigma_{xy}$$

And:

$$\sigma'_{yy} = \sum_k w_k (y'_k)^2 = \sum_k w_k \left[-sin(\theta) x_k + cos(\theta) y_k \right]^2$$

$$= sin^2(\theta) \sum_k w_k x_k^2 + cos^2(\theta) \sum_k w_k y_k^2 - 2 sin(\theta) cos(\theta) \sum_k w_k x_k y_k$$

$$\Longrightarrow \quad \sigma'_{yy} = sin^2(\theta)\sigma_{xx} + cos^2(\theta)\sigma_{yy} - 2sin(\theta)cos(\theta)\sigma_{xy}$$

Working out the latter formula somewhat further:

$$\sigma'_{yy} = \left[1 - cos^2(\theta)\right] \sigma_{xx} + \left[1 - sin^2(\theta)\right] \sigma_{yy} - 2sin(\theta)cos(\theta)\sigma_{xy}$$
$$= \sigma_{xx} + \sigma_{yy} - \sigma'_{xx}$$

It is thus seen that the sum of the two "main" moments of intertia is quite *invariant* for an orthogonal coordinate transformation:

$$\sigma'_{xx} + \sigma'_{yy} = \sigma_{xx} + \sigma_{yy}$$

We conclude that, indeed, the "unwanted" σ_{xy} can be eliminated by a suitable rotation of the coordinate system, while the sum of the other "main" second order monents ($\sigma_{xx} + \sigma_{yy}$) remains invariant.

Tensor of Inertia

In two dimensional space, the first order moments can be conceived as the two components of a vector:

$$\vec{\mu} = \begin{bmatrix} \sum_k w_k x_k \\ \sum_k w_k y_k \end{bmatrix}$$

Likewise, the second order moments can be conceived as the three components of a *symmetric matrix*, the so-called inertial tensor:

$$\overleftrightarrow{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$$

At a more sophisticated level, the problem of finding the main axes of inertia can then be approached via Linear Algebra, especially the theory of eigenvalues and eigenvectors. It's a matter of routine to show that the expressions found for the transformed moments of inertia are equivalent with the following: find the orthogonal transformation (i.e. rotation over an angle θ) which reduces the tensor of inertia to its diagonal form:

$$\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} = \begin{bmatrix} \sigma'_{xx} & 0 \\ 0 & \sigma'_{yy} \end{bmatrix}$$

This, in turn, is equivalent with finding the eigenvalues λ and the eigenvectors (κ_x, κ_y) of the inertial tensor:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \end{bmatrix} = \lambda \begin{bmatrix} \kappa_x \\ \kappa_y \end{bmatrix}$$

The corresponding characteristic equations are:

$$\begin{vmatrix} \sigma_{xx} - \lambda & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} - \lambda \end{vmatrix} = 0 \quad \Longleftrightarrow$$

$$(\sigma_{xx} - \lambda)(\sigma_{yy} - \lambda) - \sigma_{xy}^2 = 0 \quad \Longleftrightarrow \quad \lambda^2 - (\sigma_{xx} + \sigma_{yy})\lambda + \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 = 0$$

Define trace Sp and determinant Det by:

$$Sp := \sigma_{xx} + \sigma_{yy} \quad \text{and} \quad Det := \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2$$

Then the characteristic equation can be written as:

$$\lambda^2 - (Sp)\lambda + Det = 0$$

Most of the time, a quadratic equation has two solutions. The greatest of the two solutions will be called λ_1 and the smallest one λ_2. Sum and product of the solutions are found immediately:

$$\lambda_1 + \lambda_2 = Sp \quad \text{and} \quad \lambda_1.\lambda_2 = Det$$

Write the equation as:

$$\lambda^2 - 2(Sp/2)\lambda + (Sp/2)^2 = (Sp/2)^2 - Det \quad \Longleftrightarrow$$

$$[\lambda - (Sp/2)]^2 = (Sp/2)^2 - Det \quad \Longleftrightarrow \quad \lambda - (Sp/2) = \pm\sqrt{(Sp/2)^2 - Det}$$

$$\Longleftrightarrow \quad \lambda = (Sp/2) \pm \sqrt{(Sp/2)^2 - Det}$$

Herewith we find the solutions:

$$\lambda_1 = (Sp/2) + \sqrt{(Sp/2)^2 - Det}$$

$$\lambda_2 = (Sp/2) - \sqrt{(Sp/2)^2 - Det} = Det/\lambda_1$$

Provided that de discriminant is positive, indeed:

$$(Sp/2)^2 - Det = \frac{1}{4}(\sigma_{xx} + \sigma_{yy})^2 - (\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2) =$$

$$\frac{1}{4}\sigma_{xx}^2 + \frac{1}{4}\sigma_{yy}^2 + \frac{1}{2}\sigma_{xx}\sigma_{yy} - \sigma_{xx}\sigma_{yy} + \sigma_{xy}^2 = \frac{1}{4}\sigma_{xx}^2 + \frac{1}{4}\sigma_{yy}^2 - \frac{1}{2}\sigma_{xx}\sigma_{yy} + \sigma_{xy}^2 =$$

$$= \frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2 > 0$$

Herewith the eigenvalues can also be expressed as:

$$\lambda_{12} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) \pm \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\sqrt{1 + \left(\frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}\right)^2}$$

Where the expression between parentheses ( ) is recognized as:

$$\frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} = tan(2\theta)$$

The accompanying eigenvectors can now be calculated too:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \end{bmatrix} = \lambda \begin{bmatrix} \kappa_x \\ \kappa_y \end{bmatrix} \quad \Longleftrightarrow \quad \begin{cases} (\sigma_{xx} - \lambda)\kappa_x + \sigma_{xy}\kappa_y = 0 \\ \sigma_{xy}\kappa_x + (\sigma_{yy} - \lambda)\kappa_y = 0 \end{cases}$$

Possible solutions, corresponding with λ_1 and λ_2, are given by:

$$\begin{bmatrix} \kappa_x \\ \kappa_y \end{bmatrix} = \begin{bmatrix} \sigma_{xy} \\ \lambda_{12} - \sigma_{xx} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \kappa_x \\ \kappa_y \end{bmatrix} = \begin{bmatrix} \lambda_{12} - \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

For $\sigma_{xx} = \sigma_{yy}$, the eigenvalue problem reduces to:

$$(\sigma_{xx} - \lambda)^2 - \sigma_{xy}^2 = 0 \quad \Longleftrightarrow \quad \lambda = \sigma_{xx} \pm \sigma_{xy}$$

The accompanying eigenvectors are found with:

$$[\sigma_{xx} - (\sigma_{xx} \pm \sigma_{xy})]\,\kappa_x + \sigma_{xy}\kappa_y = 0 \quad \Longrightarrow$$

$$\begin{bmatrix} \kappa_x \\ \kappa_y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} / \sqrt{2} \quad \text{and} \quad \begin{bmatrix} \kappa_x \\ \kappa_y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} / \sqrt{2}$$

Since it will never happen that the length of the eigenvector belonging to one of the eigenvalues incidentally becomes zero, it is always possible to divide them by their own length. In other words: the eigenvectors may be *normed*.
It may be wondered in what circumstances the two eigenvalues of the inertial tensor become equal. This can only happen when the discriminant becomes zero:

$$(Sp/2)^2 - Det = \frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2 = 0 \quad \Longleftrightarrow$$

$$\sigma_{xx} = \sigma_{yy} \quad \text{and} \quad \sigma_{xy} = 0$$

Therefore, in this special case, the two main moments of inertia must be equal and their cross correllation must be zero. This will happen, for example, when the points cloud homogeneously fills the area of a circle.

Skewed 2-D Gaussian

It is known that the Gauss function $g(x)$ is accompanied by the following function $G(\kappa)$ as its Fourier Transform:

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}x^2/\sigma^2} \quad \Longleftrightarrow \quad G(\kappa) = \int_{-\infty}^{+\infty} g(x)\, dx = e^{-\frac{1}{2}\sigma^2\kappa^2}$$

It is seen that the standard deviation of $G(\kappa)$ in the Fourier domain is precisely the *inverse* $1/\sigma$ of the standard deviation $g(x)$ in the space domain. In the general two-dimensional case, σ^2 will be replaced by the tensor:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$$

The accompanying quadratic form is:

$$\begin{bmatrix} \kappa_x & \kappa_y \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \end{bmatrix} = \sigma_{xx}\kappa_x^2 + 2\sigma_{xy}\kappa_x\kappa_y + \sigma_{yy}\kappa_y^2$$

The Fourier Transform of a 2-D Gauss function can be generalized accordingly:

$$G(\kappa_x, \kappa_y) = e^{-\frac{1}{2}(\sigma_{xx}\kappa_x^2 + 2\sigma_{xy}\kappa_x\kappa_y + \sigma_{yy}\kappa_y^2)}$$

The exponent of the Gauss function in the Fourier domain can be brought into a more simple form by an orthogonal transformation to eigenvector coordinates. To this end, the following eigenvalue problem has been solved:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \end{bmatrix} = \lambda \begin{bmatrix} \kappa_x \\ \kappa_y \end{bmatrix}$$

Define trace Sp and determinant Det by:

$$Sp := \sigma_{xx} + \sigma_{yy} \quad \text{and} \quad Det := \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2$$

Herewith we have found the solutions:

$$\lambda_1 = (Sp/2) + \sqrt{(Sp/2)^2 - Det}$$

$$\lambda_2 = (Sp/2) - \sqrt{(Sp/2)^2 - Det} = Det/\lambda_1$$

After orthogonal transformation to a system of eigenvector coordinates, the simplified quadratic form will read as follows:

$$\lambda_1\kappa_x^2 + \lambda_2\kappa_y^2$$

Returning to the original problem, we find that the inverse $1/\sigma^2$ will correspond to the inverse of the inertia tensor:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}^{-1} = \begin{bmatrix} \sigma_{yy} & -\sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{bmatrix} /(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)$$

The accompanying quadratic form is:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \sigma_{yy} & -\sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{bmatrix} /(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2) \begin{bmatrix} x \\ y \end{bmatrix} = \frac{\sigma_{yy}x^2 - 2\sigma_{xy}xy + \sigma_{xx}y^2}{\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2}$$

This in turn corresponds to the generalization of the Gauss Function in 2-D:

$$g(x,y) = e^{-\frac{1}{2}(\sigma_{yy}x^2 - 2\sigma_{xy}xy + \sigma_{xx}y^2)/(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)}$$

A simplified quadratic form for the inverse problem can be found easily, because the eigenvalues of an inverse matrix are always the inverses of the eigenvalues of the original matrix. Hence they are found immediately to be:

$$1/\lambda_1 \quad \text{and} \quad 1/\lambda_2$$

This in turn means that the Gauss function, when transformed to eigenvector coordinates, is simply given by:

$$g(x,y) = e^{-\frac{1}{2}(x^2/\lambda_1 + y^2/\lambda_2)}$$

What's still to be found is a *norming factor* for the skewed 2-D Gaussian function. To this end, integrate the function $g(x,y)$ over the whole plane:

$$\iint g(x,y)\,dx\,dy = \iint e^{-\frac{1}{2}(x^2/\lambda_1 + y^2/\lambda_2)}\,dx\,dy$$

Substitute $u = x/\sqrt{\lambda_1}$ and $v = y/\sqrt{\lambda_2}$:

$$= \iint e^{-\frac{1}{2}(u^2+v^2)}\,d(u\sqrt{\lambda_1})\,d(v\sqrt{\lambda_2}) = \sqrt{\lambda_1\lambda_2}\iint e^{-\frac{1}{2}(u^2+v^2)}\,du\,dv =$$

Transform to polar coordinates:

$$= \sqrt{\lambda_1\lambda_2}\iint e^{-\frac{1}{2}r^2}\,rdrd\phi = -\sqrt{\lambda_1\lambda_2}\iint e^{-\frac{1}{2}r^2}\,d(-\frac{1}{2}r^2)d\phi =$$

$$= -\sqrt{\lambda_1\lambda_2}\,2\pi\left[e^{-\frac{1}{2}r^2}\right]_0^\infty = \sqrt{\lambda_1\lambda_2}\,2\pi = 2\pi\sqrt{Det}$$

Thus the norming factor for the 2-D skewed Gaussian function is, when integrated over the whole plane from $\pm\infty$ to $\pm\infty$:

$$2\pi\sqrt{\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2}$$

Giving as the end-result:

$$g(x,y) = \frac{e^{-\frac{1}{2}(\sigma_{yy}x^2 - 2\sigma_{xy}xy + \sigma_{xx}y^2)/(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)}}{2\pi\sqrt{\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2}}$$

Because Gauss functions are expensive to compute, it is desirable to have an estimate of the cut-off value, beyond which the values of Gauss function can be safely set to zero. Such is the case if, beyond certain values for (x, y):

$$e^{-\frac{1}{2}(\sigma_{yy}x^2 - 2\sigma_{xy}xy + \sigma_{xx}y^2)/(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)} < \epsilon \quad \Longleftrightarrow$$

$$-\frac{1}{2}(\sigma_{yy}x^2 - 2\sigma_{xy}xy + \sigma_{xx}y^2)/(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2) < ln(\epsilon) \quad \Longleftrightarrow$$

$$(\sigma_{yy}x^2 - 2\sigma_{xy}xy + \sigma_{xx}y^2)/(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2) > 2\,ln(1/\epsilon)$$

The outcome is far more transparent in eigenvector coordinates, where:

$$x^2/\lambda_1 + y^2/\lambda_2 > 2\,ln(1/\epsilon)$$

Substitute:

$$a^2 := \lambda_1.2\,ln(1/\epsilon)$$
$$b^2 := \lambda_2.2\,ln(1/\epsilon)$$

Then the above condition for the uninteresting area can be written as:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 > 1$$

This means that the area of interesting values for the Gauss function is bounded by an ellipse with (half) axes:

$$a := \sqrt{\lambda_1.2\,ln(1/\epsilon)}$$
$$b := \sqrt{\lambda_2.2\,ln(1/\epsilon)}$$

Hence it is possible to define a (Pascal) function for the part of the plane where values of the skewed 2-D Gauss function are worthwile to be calculated:

```
function interesting(x,y : double) : boolean;
begin
 sigma_yy*sqr(x) - 2*sigma_xy*x*y + sigma_xx*sqr(y) <
 ( sigma_xx*sigma_yy - sqr(sigma_xy) ) * 2*ln(1/epsilon)
end;
```

Most of the time, it may be necessary to have an estimate of the area which is covered by the abovementioned ellipse, for the reason that the number of points (i.e.pixels) inside the area of interest is proportional to such an area. One could, for example, carry out a *Flood Fill* on "black" pixels inside such an area. It is known that the area of an ellipse with axes a and b is given by:

$$\pi\,a.b = \pi\,\sqrt{\lambda_1.2\,ln(1/\epsilon)}\sqrt{\lambda_2.2\,ln(1/\epsilon)} = \pi\,\sqrt{\lambda_1\lambda_2}\,2\,ln(1/\epsilon) =$$

$$= ln(1/\epsilon)\,2\pi\,\sqrt{\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2}$$

Being exactly equal to the norming factor, times the logarithm of one divided by the desired accuracy. With normed functions, this leads to the incredibly simple answer that the area is equal to $ln(1/\epsilon)$.

Fuzzy Line Segment

The parameter representation of a line segment may be given by:

$$x(t) = s_x + v_x.t \quad \text{and} \quad y(t) = s_y + v_y.t \qquad \text{where:} \qquad 0 < t < T$$

Then the fuzzy line segment will be defined as follows:

$$F(x,y) = \frac{v}{\sigma\sqrt{2\pi}} \int_{t_1}^{t_2} e^{-\frac{1}{2}[x-(s_x+v_x.t)]^2/\sigma^2 - \frac{1}{2}[y-(s_y+v_y.t)]^2/\sigma^2}\, dt$$

Here the factor v is added to the norm $1/\sigma\sqrt{2\pi}$ for the sake of making the function F dimensionless: σ has the same dimension (length) as $v.dt$, if t is interpreted as time. The terms in the exponent are worked out:

$$[(s_x + v_x.t) - x]^2 + [(s_y + v_y.t) - y]^2 =$$

$$v_x^2.\,[t - (x - s_x)/v_x]^2 + v_y^2\,[(t - (y - s_y)/v_y]^2$$

Before proceeding, another sequence of formulas is presented:

$$w_1(x-A)^2 + w_2(x-B)^2 = w_1 x^2 - 2w_1 A x + w_1 A^2 + w_2 x^2 - 2w_2 B x + w_2 B^2 =$$

$$(w_1 + w_2)\left[x^2 - 2\frac{w_1 A + w_2 B}{w_1 + w_2} x + \left(\frac{w_1 A + w_2 B}{w_1 + w_2}\right)^2\right]$$

$$-\frac{(w_1 A + w_2 B)^2}{w_1 + w_2} + \frac{(w_1 A^2 + w_2 B^2)(w_1 + w_2)}{w_1 + w_2} =$$

$$(w_1 + w_2)\left[x - \frac{w_1 A + w_2 B}{w_1 + w_2}\right]^2$$

$$-\frac{w_1^2 A^2 + 2w_1 w_2 AB + w_2^2 B^2}{w_1 + w_2} + \frac{w_1^2 A^2 + w_1 w_2 A^2 + w_2^2 B^2 + w_1 w_2 B^2}{w_1 + w_2} =$$

$$(w_1 + w_2)\left[x - \frac{w_1 A + w_2 B}{w_1 + w_2}\right]^2 + \frac{w_1 w_2}{w_1 + w_2}(A^2 - 2AB + B^2)$$

The result is a *Lemma*:

$$w_1(x-A)^2 + w_2(x-B)^2 = (w_1 + w_2)\left[x - \frac{w_1 A + w_2 B}{w_1 + w_2}\right]^2 + \frac{w_1 w_2}{w_1 + w_2}(A - B)^2$$

Employ the Lemma, with $w_1 = v_x^2$, $w_2 = v_y^2$, $A = (x - s_x)/v_x$, $B = (y - s_y)/v_y$:

$$v_x^2.\,[t - (x - s_x)/v_x]^2 + v_y^2\,[(t - (y - s_y)/v_y]^2 =$$

$$(v_x^2 + v_y^2)\left[t - \frac{v_x^2.(x - s_x)/v_x + v_y^2.(y - s_y)/v_y}{v_x^2 + v_y^2}\right]^2 +$$

$$\frac{v_x^2.v_y^2}{v_x^2+v_y^2}\left[(x-s_x)/v_x-(y-s_y)/v_y\right]^2=$$

$$(v_x^2+v_y^2)\left[t-\frac{v_x.(x-s_x)+v_y.(y-s_y)}{v_x^2+v_y^2}\right]^2+\frac{\left[v_y.(x-s_x)-v_x.(y-s_y)\right]^2}{v_x^2+v_y^2}$$

First introduce a couple of abbreviations:

$$v^2=v_x^2+v_y^2 \quad \text{and} \quad \mu=\frac{v_x.(x-s_x)+v_y.(y-s_y)}{v_x^2+v_y^2}$$

Then the exponent becomes:

$$v^2(t-\mu)^2 \; + \; \left[\frac{-v_y.(x-s_x)+v_x.(y-s_y)}{v}\right]^2$$

The quotient v_x/v may be set to the cosine and the quotient v_y/v may be set to the sine of a certain angle ϕ, giving for the second term:

$$\left[-sin(\phi).(x-s_x)+cos(\phi).(y-s_y)\right]^2$$

This is recognized as the square of the function that defines a straight line, having a direction $(cos(\phi), sin(\phi))$:

$$-sin(\phi).(x-s_x)+cos(\phi).(y-s_y)=0$$

The introduction of an angle ϕ also affects the expression for μ:

$$\sqrt{v_x^2+v_y^2}\;\mu=\frac{v_x.(x-s_x)+v_y.(y-s_y)}{\sqrt{v_x^2+v_y^2}}$$

$$\Longrightarrow \quad v.\mu=cos(\phi).(x-s_x)+sin(\phi).(y-s_y)$$

$$\Longrightarrow \quad v^2(t-\mu)^2=\left[v.t-\{cos(\phi).(x-s_x)+sin(\phi).(y-s_y)\}\right]^2$$

This is recognized again as the square of the function that defines a straight line which is perpendicular to the previous one, having a normal $(cos(\phi), sin(\phi))$:

$$cos(\phi).(x-s_x)+sin(\phi).(y-s_y)=0$$

Now substitute the result into the integral:

$$F(x,y)=\frac{v}{\sigma\sqrt{2\pi}}\int_{t_1}^{t_2} e^{-\frac{1}{2}v^2(t-\mu)^2/\sigma^2+[-sin(\phi).(x-s_x)+cos(\phi).(y-s_y)]^2/\sigma^2}\,dt=$$

$$e^{-\frac{1}{2}[-sin(\phi).(x-s_x)+cos(\phi).(y-s_y)]^2/\sigma^2}\;\frac{v}{\sigma\sqrt{2\pi}}\int_{t_1}^{t_2} e^{-\frac{1}{2}v^2(t-\mu)^2/\sigma^2}\,dt$$

Thus, the exponential function splits up into a part which is quite independent on the running parameter t and another part which is still dependent on it. Only the latter has to be integrated further, of course. Due to our findings with respect to $v.\mu$, it is advantageous to introduce the *length* s as the running parameter, instead of the (time) t, by: $s = v.t$. Herewith the factor v in the norm will cancel against the factor v in $ds = dt/v$. Giving for the remaining integral:

$$I(x,y) = \frac{1}{\sigma\sqrt{2\pi}} \int_{s_1}^{s_2} e^{-\frac{1}{2}[(s-\{cos(\phi).(x-s_x)+sin(\phi).(y-s_y)\})/\sigma]^2} \, ds$$

The integral over a Gaussian can be expressed as the sum of two error functions, where the ERror Function (erf) is defined as:

$$erf(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} \, du$$

Now define as the new integration parameter:

$$u(s) = \frac{s - [cos(\phi).(x - s_x) + sin(\phi).(y - s_y)]}{\sigma}$$

Then $du = ds/\sigma$ and with $u_1 = u(s_1)$, $u_2 = u(s_2)$, we find :

$$I(x,y) = erf(u_2) - erf(u_1) \qquad \text{where} \qquad u_1 = u_1(x,y) \quad \text{and} \quad u_2 = u_2(x,y)$$

Giving as the end-result:

$$F(x,y) = [erf(u_2(x,y)) - erf(u_1(x,y))] \, e^{-\frac{1}{2}[-sin(\phi).(x-s_x)+cos(\phi).(y-s_y)]^2/\sigma^2}$$

For $(s_1, s_2) = (-\infty, +\infty)$, hence $(u_1, u_2) = (-\infty, +\infty)$, it is known that $erf(u_1) = 0$ and $erf(u_2) = 1$. Hence:

$$F(x,y) = e^{-\frac{1}{2}[-sin(\phi).(x-s_x)+cos(\phi).(y-s_y)]^2/\sigma^2}$$

Herewith we find the fuzzyfication of a straight line, with infinite length.

Conformal Mappings

In the complex plane, any straight line can be represented by the following equation:

$$z = s + t.e^{i\phi}$$

Because, when splitted into real and imaginary parts according to $z = x + i.y$ and $a = s_x + i.s_y$, this single equation is equivalent with the set:

$$x = s_x + t.sin(\phi)$$
$$y = s_y + t.cos(\phi)$$

Here t is the running parameter, which up to now is allowed to assume only real values. If the realm of the straight line is extended to the whole plane, then complex values for t should be allowed too. To indicate this, let's replace t by ζ and write:

$$z = s + \zeta.e^{i\phi}$$

Where $\zeta = \xi + i.\eta$. Solve ζ from this equation:

$$\zeta = (z - a).e^{-i\phi} = (x + i.y - s_x - i.s_y).(cos(\phi) + i.sin(\phi))$$

Resulting in:

$$\xi = Re(\zeta) = +cos(\phi).(x - s_x) + sin(\phi).(y - s_y)$$
$$\eta = Im(\zeta) = -sin(\phi).(x - s_x) + cos(\phi).(y - s_y)$$

The fuzzyfication of a straight line with tangent vector $(cos(\phi), sin(\phi))$ and infinite length is repeated:

$$F(x, y) = e^{-\frac{1}{2}[-sin(\phi).(x-s_x)+cos(\phi).(y-s_y)]^2/\sigma^2}$$

It is seen that, coincidence or not, this expression is equal to:

$$F(x, y) = e^{-\frac{1}{2}[Im(\zeta)/\sigma]^2}$$

Let's see if this result is useful for other things than straight lines. A concise equation for a circle in the complex plane is:

$$z = s + R.e^{i.\theta}$$

Where θ is the running parameter. To indicate that the equation will be declared valid for the whole complex plane, it is written as:

$$z = s + R.e^{i.\zeta}$$

Here we substitute $z = s + r.e^{i\phi}$, where $r = \sqrt{(x - s_x)^2 + (y - s_y)^2}$. Solving for ζ now gives:

$$i.\zeta = ln((z - s)/R) = ln(r/R) + i.\phi \quad \Longrightarrow$$
$$\zeta = -i.ln(r/R) + \phi \quad \Longrightarrow \quad |Im(\zeta)| = -ln(r/R)$$

An educated guess for the end-result is guided by the heuristical argument that the denominator with σ in it should be as dimensionless as the logarithm in the nominator:

$$F(x, y) = e^{-\frac{1}{2}[Im(\zeta)/\sigma]^2} = e^{-\frac{1}{2}[ln(r/R)/(\sigma/R)]^2}$$

Unsatisfied as we are with such reasoning, we will search for a relationship connecting this preliminary result to a theory wich is far better established:

the one for fuzzy straight lines. First we will make an approximation of the logarithm for large values of R:

$$ln(r/R) = ln(1 + r/R - 1) \approx r/R - 1 = \sqrt{\left(\frac{x - s_x}{R}\right)^2 + \left(\frac{y - s_y}{R}\right)^2} - 1$$

As a next step, it is recognized that the circle is approximated by its tangent line in a small area around the tangent point:

$$C - C_0 \approx \frac{\partial C}{\partial x}(x - x_0) + \frac{\partial C}{\partial y}(y - y_0)$$

Where:

$$C(x, y) = \left(\frac{x - s_x}{R}\right)^2 + \left(\frac{y - s_y}{R}\right)^2$$

And:

$$C_0 = C(x_0, y_0) = \left(\frac{x_0 - s_x}{R}\right)^2 + \left(\frac{y_0 - s_y}{R}\right)^2 = 1$$

Giving for the approximation as a whole:

$$C(x, y) \approx 1 + 2(x_0 - s_x)(x - x_0)/R^2 + 2(y_0 - s_y)(y - y_0)/R^2$$

Consequently:

$$ln(r/R) \approx \sqrt{C(x, y)} - 1 =$$

$$\sqrt{1 + 2(x_0 - s_x)(x - x_0)/R^2 + 2(y_0 - s_y)(y - y_0)/R^2} - 1$$

The square root, in turn, can be approximated as follows:

$$\sqrt{1 + x} \approx 1 + \frac{1}{2}x \quad \Longrightarrow$$

$$\sqrt{1 + 2(x_0 - s_x)(x - x_0)/R^2 + 2(y_0 - s_y)(y - y_0)/R^2} - 1$$

$$\approx 1 + (x_0 - s_x)(x - x_0)/R^2 + (y_0 - s_y)(y - y_0)/R^2 - 1 =$$

$$cos(\phi)(x - x_0)/R + sin(\phi)(y - y_0)/R$$

If we put ϕ equal to the direction of the radius R. Now we are almost there:

$$F(x, y) = e^{-\frac{1}{2}[ln(r/R)/(\sigma/R)]^2} \approx$$

$$e^{-\frac{1}{2}[\{cos(\phi)(x-x_0)/R+sin(\phi)(y-y_0)/R\}/(\sigma/R)]^2}$$

$$= e^{-\frac{1}{2}[cos(\phi).(x-s_x)+sin(\phi).(y-s_y)]^2/\sigma^2}$$

Almost, because there seems to be a discrepancy between this formula and the one for the fuzzy straight line:

$$e^{-\frac{1}{2}[-sin(\phi).(x-s_x)+cos(\phi).(y-s_y)]^2/\sigma^2}$$

This seemingly paradoxal result is readily resolved by recognizing that the *normal* vector of the tangent line has been employed with the limiting case of our circle fuzzyfication, while the *direction* of the straight line itself was used in the earlier result. Thus we only have to replace the normal $cos(\phi), sin(\phi)$ by the direction $-sin(\phi), cos(\phi)$ and the fuzzyfication of the infinite straight line will be found back, indeed. This also means that the heuristics, which gave rise to the factor (σ/R) , is justified, in the end:

$$F(x,y) = e^{-\frac{1}{2}[ln(r/R)/(\sigma/R)]^2} \qquad \text{where:} \qquad r = \sqrt{(x-s_x)^2 + (y-s_y)^2}$$

Because Gauss functions are expensive to compute, it is desirable to have an estimate of the cut-off value, beyond which the values of Gauss function can be safely set to zero. Such is the case if, beyond certain values for r:

$$e^{-\frac{1}{2}[ln(r/R)/(\sigma/R)]^2} < \epsilon \quad \Longrightarrow \quad [ln(r/R)/(\sigma/R)]^2 > 2\,ln(1/\epsilon) \quad \Longrightarrow$$

$$|ln(r/R)| > \frac{\sigma}{R}\sqrt{2\,ln(1/\epsilon)}$$

Consequently, the interesting values of r are restricted to the interval:

$$R.e^{-\sqrt{2\,ln(1/\epsilon)}\;\sigma/R} \le r \le R.e^{+\sqrt{2\,ln(1/\epsilon)}\;\sigma/R}$$

With the *standard accuracy* $\epsilon = e^{-\frac{1}{2}(2\pi)^2}$, which I have introduced elswhere, the formula becomes:

$$R.e^{-2\pi\sigma/R} \le r \le R.e^{+2\pi\sigma/R}$$