Lie Groups 1-D

Any differentiable function can be developed into a Taylor series around one of its points:

$$f(x+a) = f(a) + x.f'(a) + \frac{1}{2}x^2.f''(a) + ...$$

Here the choice of the point a is arbitrary. A Taylor series expansion around point zero is quite common:

$$f(x) = f(0) + x.f'(0) + \frac{1}{2}x^2.f''(0) + ...$$

But, the other way around, one could also write:

$$f(x+a) = f(x) + a.f'(x) + \frac{1}{2}a^2.f''(x) + ...$$

With physical systems, there is no preferred origin. This means that, always, a function f_1 can be defined, which is merely shifted with respect to f with a certain amount a :

$$f_1(x) = f(x+a) = f(x) + a.f'(x) + \frac{1}{2}a^2.f''(x) + \frac{1}{3!}a^3 f'''(x) + ...$$

We thus see that the shifted *function* may be expanded as a Taylor series around the original function. It is emphasized that developing a function as a power series around another function is *quite different* from developing a function as a power series around one of its own values, though the formulas are more or less similar. The series expansion of the shifted function can also be written as follows, using knowledge from the preceding paragraphs:

$$f_1(x) = f(x+a) = e^{a.\frac{d}{dx}} f(x)$$

Thus the shifted function is obtained by taking into account the effect of the *translation operator* $exp(a.d/dx)$ as it acts upon $f(x)$.
Instead of translating a function over a certain distance, which seems to be a rather trivial operation anyway, let us consider *scaling*. This means that we are going to make intervals of the independent variable smaller, or larger, with a factor $\lambda > 0$. The transformed function is then defined by:

$$f_1(x) = f(\lambda.x)$$

Like with translations, it would be nice to develop the function $f_1(x)$ into a Taylor series expansion around the original $f(x)$. But this is not as simple as in the former case. Unless a clever trick is devised, which reads as follows. Define a couple of new variables, a and y, and a new function g:

$$\lambda = e^a \quad \text{and} \quad x = e^y \quad \text{and} \quad g(y) = f(e^y)$$

Then, indeed, we can develop something into a Taylor series:

$$f_1(x) = f(e^a.e^y) = f(e^{a+y}) = g(y+a) = e^{a.\frac{d}{dy}}g(y)$$

A variable such as y, which renders the transformation to be resemblant to a translation, is commonly called a *canonical* variable. In the case of a scaling transformation, the canonical variable is obtained by taking the logarithm of the independent variable: $y = ln(x)$. Working back to the original variables and the original function:

$$g(y) = f(e^y) = f(x) \quad \text{and} \quad a = ln(\lambda)$$

$$\frac{d}{dy} = \frac{dx}{dy}\frac{d}{dx} = e^y\frac{d}{dx} = x\frac{d}{dx}$$

The operator $x.d/dx$ is commonly called the *infinitesimal operator* of a scaling transformation. An infinitesimal operator always equals differentiation to the canonical variable, which converts the transformation into a translation. We have already met, of course, the infinitesimal operator for the translations themselves, which is simply given by (d/dx). This all leads, rather quickly, to the following somewhat less-trivial result:

$$f_1(x) = f(\lambda.x) = e^{ln(\lambda).x\frac{d}{dx}}f(x)$$

Written out as a "true" Taylor series:

$$f(\lambda.x) = f(x) + ln(\lambda).x\frac{df}{dx} + \frac{1}{2}ln^2(\lambda).x\frac{d(x.df/dx)}{dx} + ...$$

However, there is something bogus about this argument, since it is obvious that the result has only be validated here for $x > 0$. In order to be certain that things are also valid for negative values of x , we must actually carry out the calculation. It is sufficient to do this for the scaling transformation of x itself, which is represented by the series $exp(ln(\lambda).x.d/dx)x$:

$$e^{ln(\lambda).x\frac{d}{dx}}x = x + ln(\lambda).x\frac{dx}{dx} + \frac{1}{2}ln^2(\lambda).x\frac{d(x.dx/dx)}{dx} + ...$$

$$= \left[1 + ln(\lambda) + \frac{1}{2}ln^2(\lambda) + ...\right]x = e^{ln(\lambda)}x = \lambda x$$

Herewith the series expansion is verified for all real values of x . However, since λ must be positive, there exists no continuous transition towards problems where values are, at the same time, inverted or *mirrorred*, like in:

$$f_1(x) = f(-\lambda.x)$$

For this to happen, the scaling transformation would to have to pass through a point where everything is contracted to zero:

$$f_1(x) = f(0.x)$$

This already reveals a glimpse of the *topological issues* which may be associated with the notion of Taylor series around a function. These have been troubling the area of research ever since. (Quite unneccessarily in my opinion, but that's another matter.)

Lie Groups 2-D

An example of a Continuous transformation in two dimensions is Rotation over an angle θ:

$$\begin{cases} x_1 = cos(\theta).x - sin(\theta).y \\ y_1 = sin(\theta).x + cos(\theta).y \end{cases}$$

It might be asked how the rotation of the independent variables works out for a function of these variables. With other words, how the following function would be expanded as a Taylor series expansion around the original $f(x, y)$:

$$f_1(x, y) = f(x_1, y_1) = f(cos(\theta).x - sin(\theta).y, sin(\theta).x + cos(\theta).y)$$

Define other (polar) variables (r, ϕ) as:

$$x = r.cos(\phi) \quad \text{and} \quad y = r.sin(\phi)$$

Giving for the transformed variables:

$$x_1 = r.cos(\phi).cos(\theta) - r.sin(\phi).sin(\theta) = r.cos(\phi + \theta)$$
$$y_1 = r.cos(\phi).sin(\theta) + r.sin(\phi).cos(\theta) = r.sin(\phi + \theta)$$

We see that ϕ is a proper canonical variable. Another function $g(\psi)$ is defined with this canonical variable as the independent one:

$$g(\phi) = f(r.cos(\phi), r.sin(\phi)) = f(x, y)$$

Now rotating $f(x, y)$ over an angle θ corresponds with a translation of $g(\phi)$ over a distance θ. Therefore $g(\phi + \theta)$ can be developed into a Taylor series around the original:

$$g(\phi + \theta) = g(\phi) + \theta.\frac{dg(\phi)}{d\phi} + \frac{1}{2}\theta^2.\frac{d^2g}{d\phi^2} + ...$$

Working back to the original variables (x, y) with a well known chain rule for partial derivatives:

$$\frac{dg}{d\phi} = \frac{\partial g}{\partial x}\frac{dx}{d\phi} + \frac{\partial g}{\partial y}\frac{dy}{d\phi}$$

Where:

$$\frac{dx}{d\phi} = -r.sin(\phi) = -y \quad \text{and} \quad \frac{dy}{d\phi} = +r.cos(\phi) = +x \quad \Longrightarrow$$

$$\frac{dg}{d\phi} = \frac{\partial g}{\partial x}.(-y) + \frac{\partial g}{\partial y}.(+x) \quad \Longrightarrow \quad \frac{d}{d\phi} = x.\frac{\partial}{\partial y} - y.\frac{\partial}{\partial x}$$

Herewith we find that the operator $(x.\frac{\partial}{\partial y} - y.\frac{\partial}{\partial x})$ is the *infinitesimal operator* for Plane Rotations. It is equal to differentiation with respect to the canonical variable, as expected. The end-result is:

$$f_1(x,y) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\theta(x.\frac{\partial}{\partial y} - y.\frac{\partial}{\partial x}) \right]^k f(x,y) = e^{\theta(x\partial/\partial y - y\partial/\partial x)} f(x,y)$$

This is true for *any* function $f(x,y)$. In particular, the independent variables themselves can be conceived as such functions. Which means that:

$$x_1 = e^{\theta(x\partial/\partial y - y\partial/\partial x)} x \quad \text{and} \quad y_1 = e^{\theta(x\partial/\partial y - y\partial/\partial x)} y$$

It is easily demonstrated that:

$$(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})x = -y \quad \text{and} \quad (x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})y = x$$

Herewith we find:

$$\sum_{k=0}^{\infty} \left[\theta(x.\frac{\partial}{\partial y} - y.\frac{\partial}{\partial x}) \right]^k x = 1 - \theta.y - \frac{1}{2}\theta^2.x + \frac{1}{3!}\theta^3.y + \frac{1}{4!}\theta^4.x + ...$$

$$= cos(\theta).x - sin(\theta).y = x_1$$

Likewise we find:

$$\sum_{k=0}^{\infty} \left[\theta(x.\frac{\partial}{\partial y} - y.\frac{\partial}{\partial x}) \right]^k y = 1 + \theta.x - \frac{1}{2}\theta^2.y - \frac{1}{3!}\theta^3.x + \frac{1}{4!}\theta^4.y + ...$$

$$= sin(\theta).x + cos(\theta).y = y_1$$

Thus, indeed, the formulas for a far-form-infinitesimal rotation over an finite angle θ can be reconstructed from the expansions.
Now suppose that we change a minor thing in the infinitesimal operator for plane rotations, namely the minus sign into a plus sign, giving:

$$\frac{d}{dp} = x.\frac{\partial}{\partial y} + y.\frac{\partial}{\partial x}$$

And it is wondered what global transformation will come out of such a subtle change. Calculate in very much the same way:

$$x_1 = \sum_{k=0}^{\infty} \left[p(x.\frac{\partial}{\partial y} + y.\frac{\partial}{\partial x}) \right]^k x \quad \text{and} \quad y_1 = \sum_{k=0}^{\infty} \left[p(x.\frac{\partial}{\partial y} + y.\frac{\partial}{\partial x}) \right]^k y$$

It is easily demonstrated that:

$$(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x})x = y \quad \text{and} \quad (x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x})y = x$$

Herewith we find:

$$\sum_{k=0}^{\infty} \left[p(x.\frac{\partial}{\partial y} + y.\frac{\partial}{\partial x}) \right]^k x = 1 + p.y + \frac{1}{2}p^2.x + \frac{1}{3!}p^3.y + \frac{1}{4!}p^4.x + ...$$

$$= cosh(p).x + sinh(p).y = x_1$$

Likewise we find:

$$\sum_{k=0}^{\infty} \left[p(x.\frac{\partial}{\partial y} + y.\frac{\partial}{\partial x}) \right]^k y = 1 + p.x + \frac{1}{2}p^2.y + \frac{1}{3!}p^3.x + \frac{1}{4!}p^4.y + ...$$

$$= sinh(p).x + cosh(p).y = y_1$$

For those who don't know what the meaning is of a hyperbolic cosine $cosh(p)$ and a hyperbolic sine $sinh(p)$, the definitions are:

$$cosh(p) = \frac{e^{+p} + e^{-p}}{2} \quad \text{and} \quad sinh(p) = \frac{e^{+p} - e^{-p}}{2}$$

The truth of the above can easily be checked out now. What we have found here is the following transformation:

$$x_1 = cosh(p).x + sinh(p).yy_1 = sinh(p).x + cosh(p).y$$

In structural mechanics, this beasty is known as the *Deformation Tensor* and, as such, it should have been known for a some time. But it really became a famous transformation when the theory of Special Relativity was discovered, where it is known as the Lorentz Transformation. A special property of pure deformations, like with rotations, is that the determinant of the matrix is always equal to unity:

$$\left\| \begin{array}{cc} cosh(p) & sinh(p) \\ sinh(p) & cosh(p) \end{array} \right\| = cosh^2(p) - sinh^2(p) = 1$$

Meaning that pure deformations leave volumes of the material unchanged.
This more or less completes our discussion about transformations in one and

two dimensions. We have found six infinitesimal operators in total, if we also take into account both one-dimensional (x, y) directions in the plane. Here comes a list of differentiations to the canonical variables, also called infinitesimal transformations, which have been found so far.

$$\begin{array}{rc} \text{Translation x:} & \partial/\partial x \\ \text{Translation y:} & \partial/\partial y \\ \text{Scaling for x:} & \partial/\partial ln(x) = x.\partial/\partial x \\ \text{Scaling for y:} & \partial/\partial ln(y) = y.\partial/\partial y \\ \text{Flat rotation:} & d/d\theta = x.\partial/\partial y - y.\partial/\partial x \\ \text{Flat skewness:} & d/dp = x.\partial/\partial y + y.\partial/\partial x \end{array}$$

What we are talking about here all the time is the so-called *Theory of Lie Groups*. Most of the time, Lie Groups have been the kind of mathematics which tend to be incomprehensible for most human beings. Maybe that's because it is embedded, nowadays, a great deal, in Physics on Elementary Particles and other SF-like subjects. It has been shown here that very basic, 1- and 2-dimensional examples provide useful material to demonstrate the whole idea, *the gist* of the Theory, while the more difficult, "advanced" topics can be left aside.