Chebyshev and stuff

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Chebyshev Polynomials have shown up in my homework as a by-product of Cosine Expansions and DoubleGrid - TripleGrid Calculus:

http://hdebruijn.soo.dto.tudelft.nl/hdb_spul/cospower.pdf
http://hdebruijn.soo.dto.tudelft.nl/hdb_spul/calculus.pdf
http://hdebruijn.soo.dto.tudelft.nl/jaar2006/drievoud.pdf

Further musings on the subject have been motivated by a poster in the Usenet group 'sci.math.num-analysis':

http://groups.google.nl/group/sci.math.num-analysis/msg/24d6a06635f30b5a

The technique to be employed preferrably for solving differential equations is known as Heaviside's Operational Calculus, also called *Operator Calculus*. This Operator Calculus is an absolute prerequisite for everything that follows here. The next article provides a lucid exposition of the methods involved:

http://hdebruijn.soo.dto.tudelft.nl/jaar2004/uitboek.pdf

Special Solutions

The differential equation named after Pafnuty Chebyshev is:

$$(1-x^2)\frac{d^2T_n(x)}{dx^2} - x\frac{dT_n(x)}{dx} + n^2T_n(x) = 0$$

When cast in Operator Calculus format, it reads:

$$\left[(1-x^2)\left(\frac{d}{dx}\right)^2 - x\frac{d}{dx} + n^2\right]T_n(x) = 0$$

We shall try to find a factorization:

$$\left[(1-x^2)\left(\frac{d}{dx}\right)^2 - x\frac{d}{dx} + n^2\right] = \left[(1+x)\frac{d}{dx} + \alpha\right]\left[(1-x)\frac{d}{dx} + \beta\right]$$

Or alternatively:

$$\left[(1-x^2)\left(\frac{d}{dx}\right)^2 - x\frac{d}{dx} + n^2\right] = \left[(1-x)\frac{d}{dx} + \alpha\right]\left[(1+x)\frac{d}{dx} + \beta\right]$$

Working out the first alternative, with the rule $d/dxf = fd/dx + f'$:

$$\left[(1+x)\frac{d}{dx} + \alpha\right]\left[(1-x)\frac{d}{dx} + \beta\right] =$$

$$(1+x)\frac{d}{dx}(1-x)\frac{d}{dx}+\alpha(1-x)\frac{d}{dx}+\beta(1+x)\frac{d}{dx}+\alpha\beta =$$

$$(1-x^2)\left(\frac{d}{dx}\right)^2+\left[-(1+x)+\alpha(1-x)+\beta(1+x)\right]\frac{d}{dx}+\alpha\beta =$$

$$\left[(1-x^2)\frac{d^2}{dx^2}-x\frac{d}{dx}+n^2\right]$$

It follows that:

$$-(1+x)+\alpha(1-x)+\beta(1+x)=(-1+\alpha+\beta)+(-1-\alpha+\beta)x=-x$$

$$\iff \quad \alpha+\beta=1 \quad \text{and} \quad \alpha=\beta \quad \iff \quad \alpha=\beta=\frac{1}{2}$$

Working out the second alternative:

$$\left[(1-x)\frac{d}{dx}+\alpha\right]\left[(1+x)\frac{d}{dx}+\beta\right]=$$

$$(1-x)\frac{d}{dx}(1+x)\frac{d}{dx}+\alpha(1+x)\frac{d}{dx}+\beta(1-x)\frac{d}{dx}+\alpha\beta =$$

$$(1-x^2)\left(\frac{d}{dx}\right)^2+\left[+(1-x)+\alpha(1+x)+\beta(1-x)\right]\frac{d}{dx}+\alpha\beta =$$

$$\left[(1-x^2)\frac{d^2}{dx^2}-x\frac{d}{dx}+n^2\right]$$

It follows that:

$$+(1-x)+\alpha(1+x)+\beta(1-x)=(1+\alpha+\beta)+(-1+\alpha-\beta)x=-x$$

$$\iff \quad \alpha+\beta=-1 \quad \text{and} \quad \alpha=\beta \quad \iff \quad \alpha=\beta=-\frac{1}{2}$$

Thus, with such a factorization, only a quite special case of the differential equation can be handled:

$$\left[(1-x^2)\frac{d^2}{dx^2}-x\frac{d}{dx}+n^2\right]T_n(x)=0 \qquad \text{where} \quad n=\pm\frac{1}{2}$$

The following may be considered as the Main Formula of Operator Calculus:

$$\boxed{\frac{d}{dx}+f=e^{-\int f\,dx}\,\frac{d}{dx}\,e^{+\int f\,dx}}$$

With help this formula, we find for the different factors of the very special differential equation by Chebyshev:

$$\left[(1-x)\frac{d}{dx}+\frac{1}{2}\right]=(1-x)\left[\frac{d}{dx}+\frac{1}{2}\frac{1}{1-x}\right]=$$

$$(1-x)\; e^{-\int 1/(2(1-x))\, dx}\; \frac{d}{dx}\; e^{+\int 1/(2(1-x))\, dx} =$$

$$(1-x)\; e^{ln(1-x)/2}\; \frac{d}{dx}\; e^{-ln(1-x)/2} \quad \Longrightarrow$$

$$\left[(1-x)\frac{d}{dx}+\frac{1}{2}\right] = (1-x)\; \sqrt{1-x}\; \frac{d}{dx}\; \frac{1}{\sqrt{1-x}}$$

And in very much the same way:

$$\left[(1-x)\frac{d}{dx}-\frac{1}{2}\right] = (1-x)\; \frac{1}{\sqrt{1-x}}\; \frac{d}{dx}\; \sqrt{1-x}$$

$$\left[(1+x)\frac{d}{dx}+\frac{1}{2}\right] = (1+x)\; \frac{1}{\sqrt{1+x}}\; \frac{d}{dx}\; \sqrt{1+x}$$

$$\left[(1+x)\frac{d}{dx}-\frac{1}{2}\right] = (1+x)\; \sqrt{1+x}\; \frac{d}{dx}\; \frac{1}{\sqrt{1+x}}$$

Thus there are two ways of solving:

$$\left[(1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx} + \frac{1}{4}\right] T_{+1/2}(x) =$$

$$\left[(1+x)\frac{d}{dx}+\frac{1}{2}\right]\left[(1-x)\frac{d}{dx}+\frac{1}{2}\right] T_{+1/2}(x) =$$

$$\left[(1+x)\; \frac{1}{\sqrt{1+x}}\; \frac{d}{dx}\; \sqrt{1+x}(1-x)\; \sqrt{1-x}\; \frac{d}{dx}\; \frac{1}{\sqrt{1-x}}\right] T_{+1/2}(x) = 0$$

$$\Longrightarrow \quad (1-x)\sqrt{1-x^2}\; \frac{d}{dx}\; \frac{1}{\sqrt{1-x}}\; T_{+1/2}(x) = C$$

$$\Longrightarrow \quad T_{+1/2}(x) = C\sqrt{1-x}\int \frac{dx}{(1-x)\sqrt{1-x^2}}$$

On the other hand:

$$\left[(1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx} + \frac{1}{4}\right] T_{-1/2}(x) =$$

$$\left[(1-x)\frac{d}{dx}-\frac{1}{2}\right]\left[(1+x)\frac{d}{dx}-\frac{1}{2}\right] T_{-1/2}(x) =$$

$$\left[(1-x)\; \frac{1}{\sqrt{1-x}}\; \frac{d}{dx}\; \sqrt{1-x}(1+x)\; \sqrt{1+x}\; \frac{d}{dx}\; \frac{1}{\sqrt{1+x}}\right] T_{-1/2}(x) = 0$$

$$\Longrightarrow \quad (1+x)\sqrt{1-x^2}\; \frac{d}{dx}\; \frac{1}{\sqrt{1+x}}\; T_{-1/2}(x) = C$$

$$\Longrightarrow \quad T_{-1/2}(x) = C\sqrt{1+x}\int \frac{dx}{(1+x)\sqrt{1-x^2}}$$

It certainly helps to know the following facts, with $(u/v)' = (u'v - v'u)/v^2$ and $d\sqrt{1 \pm x}/dx = \pm 1/\sqrt{1 \pm x}$:

$$\frac{d}{dx}\left(\sqrt{\frac{1+x}{1-x}}\right) = \frac{\sqrt{1-x}/\sqrt{1+x} + \sqrt{1+x}/\sqrt{1-x}}{1-x} =$$

$$\frac{1-x+1+x}{\sqrt{1-x}\sqrt{1+x}(1-x)} = \frac{1}{(1-x)\sqrt{1-x^2}}$$

$$\Longrightarrow \quad \int \frac{dx}{(1-x)\sqrt{1-x^2}} = \sqrt{\frac{1+x}{1-x}}$$

$$\Longrightarrow \quad T_{+1/2}(x) = C\sqrt{1-x}\left[\sqrt{\frac{1+x}{1-x}} + D\right] = A\sqrt{1-x} + B\sqrt{1+x}$$

$$\frac{d}{dx}\left(\sqrt{\frac{1-x}{1+x}}\right) = \frac{-\sqrt{1+x}/\sqrt{1-x} - \sqrt{1-x}/\sqrt{1+x}}{1+x} =$$

$$\frac{-1+x-1-x}{\sqrt{1-x}\sqrt{1+x}(1+x)} = \frac{1}{(1+x)\sqrt{1-x^2}}$$

$$\Longrightarrow \quad \int \frac{dx}{(1+x)\sqrt{1-x^2}} = \sqrt{\frac{1-x}{1+x}}$$

$$\Longrightarrow \quad T_{-1/2}(x) = C\sqrt{1+x}\left[\sqrt{\frac{1-x}{1+x}} + D\right] = A\sqrt{1-x} + B\sqrt{1+x}$$

Where A and B are arbitrary integration constants. It is concluded that the general form of a special solution of the Chebyshev differential equation is:

$$T_{-1/2}(x) = T_{+1/2}(x) = A\sqrt{1-x} + B\sqrt{1+x}$$

Ladder Operators

The following is essentially a elaboration of *Chebyshev polynomials*, which is a rather terse paragraph 3.4 in:

`http://www.smf.mx/rmf/pdf/rmf/49_4/49_358.pdf`

The big trick is to write the Chebyshev differential equation in a slightly different form, multiplied namely with a factor $(1-x^2)$. Then the very same differential equation named after Pafnuty Chebyshev becomes:

$$(1-x^2)^2\frac{d^2T_n(x)}{dx^2} - (1-x^2)x\frac{dT_n(x)}{dx} + (1-x^2)n^2T_n(x) = 0$$

When cast in Operator Calculus format, it reads:

$$\left[(1-x^2)^2\left(\frac{d}{dx}\right)^2-(1-x^2)x\frac{d}{dx}+(1-x^2)n^2\right]T_n(x)=0$$

We shall try to find a factorization again. Be careful:

$$\left[(1-x^2)^2\left(\frac{d}{dx}\right)^2-(1-x^2)x\frac{d}{dx}+(1-x^2)n^2\right]=$$

$$\left[(1-x^2)\frac{d}{dx}+\alpha\,x\right]\left[(1-x^2)\frac{d}{dx}+\beta\,x\right]=$$

$$(1-x^2)\frac{d}{dx}(1-x^2)\frac{d}{dx}+\alpha\,x(1-x^2)\frac{d}{dx}+(1-x^2)\frac{d}{dx}\beta\,x+\alpha\,x\beta\,x=$$

$$(1-x^2)^2\frac{d^2}{dx^2}+$$

$$\left[-2x(1-x^2)+\alpha\,x(1-x^2)+(1-x^2)\beta\,x\right]\frac{d}{dx}+$$

$$(1-x^2)\beta+\alpha\beta\,x^2$$

Where:

$$-2x(1-x^2)+\alpha\,x(1-x^2)+(1-x^2)\beta\,x=-x(1-x^2)$$

$$\Longrightarrow\quad -2+\alpha+\beta=-1\quad\Longrightarrow\quad \beta=1-\alpha\quad\text{and}\quad\alpha=1-\beta$$

And:

$$(1-x^2)\beta+\alpha\beta\,x^2=(1-x^2)\beta-\alpha\beta(1-x^2)+\alpha\beta=$$

$$(1-x^2)\beta(1-\alpha)+\alpha\beta=(1-x^2)\beta^2+\alpha\beta=(1-x^2)n^2$$

If we put $\beta=+n$ then $\alpha=-(n-1)$ and a factor $-\alpha\beta=n(n-1)$ must be added to the the left and to the right hand side:

$$\left[(1-x^2)\frac{d}{dx}-(n-1)\,x\right]\left[(1-x^2)\frac{d}{dx}+n\,x\right]T_n(x)=n(n-1)T_n(x)$$

If we put $\beta=-n$ then $\alpha=(n+1)$ and a factor $-\alpha\beta=n(n+1)$ must be added to the the left and to the right hand side:

$$\left[(1-x^2)\frac{d}{dx}+(n+1)\,x\right]\left[(1-x^2)\frac{d}{dx}-n\,x\right]T_n(x)=n(n+1)T_n(x)$$

Now define the following operator:

$$O_m=\left[(1-x^2)\frac{d}{dx}+m\,x\right]$$

Then the differential equation by Chebyshev assumes one of the following forms:

$$\begin{aligned} O_{-(n-1)}\, O_{+n}\, T_n(x) &= n(n-1)\, T_n(x) \\ O_{+(n+1)}\, O_{-n}\, T_n(x) &= n(n+1)\, T_n(x) \end{aligned}$$

Multiply the first of these two equations on the left with the operator O_{+n}, times an arbitrary constant c_1 eventually:

$$c_1\, O_{+n} \left[O_{-(n-1)}\, O_{+n}\right] T_n(x) = c_1\, O_{+n}\, n(n-1) T_n(x)$$

$$\left[O_{+(n-1)+1}\, O_{-(n-1)}\right] c_1\, O_{+n}\, T_n(x) = n(n-1)\, c_1\, O_{+n}\, T_n(x)$$

Which demonstrates that $c_1\, O_{+n}\, T_n(x)$ is a solution of the differential equation for $n := n-1$. This means that, effectively:

$$c_1\, O_{+n}\, T_n(x) = T_{n-1}(x) = c_1 \left[(1-x^2)\frac{d}{dx} + n\,x\right] T_n(x)$$

For this reason, the operator O_{+n} is called a *lowering operator*.
Multiply the second of the two equations on the left with the operator O_{-n}, times an arbitrary constant c_2 eventually:

$$c_2\, O_{-n} \left[O_{+(n+1)}\, O_{-n}\right] T_n(x) = c_2\, O_{-n}\, n(n+1) T_n(x)$$

$$\left[O_{-(n+1)+1}\, O_{+(n+1)}\right] c_2\, O_{-n}\, T_n(x) = n(n+1)\, c_2\, O_{-n}\, T_n(x)$$

Which demonstrates that $c_2\, O_{-n}\, T_n(x)$ is a solution of the differential equation for $n := n+1$. This means that, effectively:

$$c_2\, O_{-n}\, T_n(x) = T_{n+1}(x) = c_2 \left[(1-x^2)\frac{d}{dx} - n\,x\right] T_n(x)$$

For this reason, the operator O_{-n} is called a *raising operator*. The raising and lowering operators together are called *ladder operators*, because they enable us to construct a whole sequence of solutions, once we have found only one of the possible ones. Let's repeat the result:

$$\begin{aligned} c_1 \left[(1-x^2)\frac{d}{dx} - n\,x\right] T_n(x) &= T_{n+1}(x) \\ c_2 \left[(1-x^2)\frac{d}{dx} + n\,x\right] T_n(x) &= T_{n-1}(x) \end{aligned}$$

If we add these equations together, then:

$$T_{n+1}(x) + T_{n-1}(x) = [-nc_1 + nc_2]\, x\, T_n(x) + [c_1 + c_2]\,(1-x^2)\frac{dT_n}{dx}$$

While the well known recursion relation for Chebyshev Polynomials is:

$$T_{n+1}(x) + T_{n-1}(x) = 2x\, T_n(x)$$

So if we put the arbitrary constants $c_{1,2}$ to well defined values, namely $c_1 = -1/n$ and $c_2 = +1/n$, then the recursion relation for solutions of the Chebyshev differential equation becomes the same as the one for Chebyshev polynomials:

$$T_{n+1}(x) + T_{n-1}(x) = 2x\, T_n(x)$$

And the ladder relations become exactly as in paragraph 3.4 of the paper on *One-parameter isospectral special functions* :

$$\begin{aligned}\left[(1-x^2)\frac{d}{dx} - n\,x\right] T_n(x) &= -n\, T_{n+1}(x) \\ \left[(1-x^2)\frac{d}{dx} + n\,x\right] T_n(x) &= +n\, T_{n-1}(x)\end{aligned}$$

Whole Integer Solutions

The differential equation named after Pafnuty Chebyshev is:

$$(1-x^2)\frac{d^2T_n(x)}{dx^2} - x\frac{dT_n(x)}{dx} + n^2T_n(x) = 0$$

When cast in Operator Calculus format, it reads:

$$\left[(1-x^2)\left(\frac{d}{dx}\right)^2 - x\frac{d}{dx} + n^2\right] T_n(x) = 0$$

We shall try to find a factorization for $n = 0$:

$$\left[(1-x^2)\left(\frac{d}{dx}\right)^2 - x\frac{d}{dx}\right] T_0(x) = 0$$

$$\left[(1-x^2)\frac{d}{dx} - x\right]\frac{d}{dx}T_0(x) = 0$$

The following may be considered as the Main Formula of Operator Calculus:

$$\frac{d}{dx} + f = e^{-\int f\,dx}\,\frac{d}{dx}\,e^{+\int f\,dx}$$

With help this formula, we find for the main factor of Chebyshev's differential equation for $n = 0$:

$$(1-x^2)\frac{d}{dx} - x = (1-x^2)\left[\frac{d}{dx} + \frac{1}{2}\frac{-2x}{1-x^2}\right] =$$

$$(1-x^2)\exp\left[-\frac{1}{2}\int\frac{d(1-x^2)}{1-x^2}\right]\,\frac{d}{dx}\,\exp\left[+\frac{1}{2}\int\frac{d(1-x^2)}{1-x^2}\right]$$

Herewith we can complete the sequence of formulas leading to the solution:

$$(1-x^2)\frac{1}{\sqrt{1-x^2}}\,\frac{d}{dx}\,\sqrt{1-x^2}\,\frac{d}{dx}\,T_0(x) = 0 \quad \Longrightarrow$$

$$\sqrt{1-x^2}\,\frac{d}{dx}\,T_0(x) = B \quad \Longrightarrow$$

$$T_0(x) = B\int\frac{dx}{\sqrt{1-x^2}} = A + B\arccos(x)$$

Where A and B are arbitrary integration constants.
In order to find a clue for determining the solutions $T_1(x)$ for $n = 1$, we take a closer look at the first operator representation of Chebyshev's differential equation, as has been derived in the paragraph *Ladder Operators*:

$$\left[(1-x^2)\frac{d}{dx} - (n-1)\,x\right]\left[(1-x^2)\frac{d}{dx} + n\,x\right]T_n(x) = n(n-1)T_n(x)$$

In our case $n = 1$, therefore:

$$(1-x^2)\frac{d}{dx}\left[(1-x^2)\frac{d}{dx} + x\right]T_1(x) = 0$$

$$\Longrightarrow \quad \left[(1-x^2)\frac{d}{dx} + x\right]T_1(x) = C$$

Employing almost the same procedure as above, we find:

$$(1-x^2)\frac{d}{dx} + x = (1-x^2)\left[\frac{d}{dx} - \frac{1}{2}\frac{-2x}{1-x^2}\right] =$$

$$(1-x^2)\exp\left[+\frac{1}{2}\int\frac{d(1-x^2)}{1-x^2}\right]\,\frac{d}{dx}\,\exp\left[-\frac{1}{2}\int\frac{d(1-x^2)}{1-x^2}\right]$$

And we can complete the sequence of formulas leading to the solution:

$$(1-x^2)\sqrt{1-x^2}\,\frac{d}{dx}\,\frac{1}{\sqrt{1-x^2}}\,T_1(x) = C \quad \Longrightarrow$$

$$T_1(x) = C\sqrt{1-x^2}\left[\int\frac{dx}{(1-x^2)\sqrt{1-x^2}}\right]$$

The solution is:

$$T_1(x) = C\sqrt{1-x^2}\left[\frac{x}{\sqrt{1-x^2}} + D/C\right] = Cx + D\sqrt{1-x^2}$$

Where C and D are arbitrary integration constants.
Time to repeat the ladder relations:

$$\begin{aligned}\left[(1-x^2)\frac{d}{dx} - n\,x\right]T_n(x) &= -n\,T_{n+1}(x)\\ \left[(1-x^2)\frac{d}{dx} + n\,x\right]T_n(x) &= +n\,T_{n-1}(x)\end{aligned}$$

When specified for $n = 0$, the first ladder relation reads:

$$\left[(1-x^2)\frac{d}{dx}\right] T_0(x) = 0 \quad \Longleftrightarrow \quad T_0(x) = A$$

Where A is an arbitrary integration constant. Thus we see that $B\arccos(x)$ inevitably must drop out of the ladder solutions. But let's see what happens, nevertheless, if we don't do this. With the recurrence relation for Chebyshev functions we find:

$$T_2(x) = 2x\,T_1(x) - T_0(x) = 2x\left[Cx + D\sqrt{1-x^2}\right] - A - B\arccos(x)$$

When substituted into the Chebyshev differential equation for $n = 2$:

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> simplify((1-x^2)*diff(diff(T2(x),x),x)-x*diff(T2(x),x)+2^2*T2(x));
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$$4C - 4A - 4B\arccos(x)$$

Indeed, we can only force the outcome to zero if $B = 0$ and if $A = C$:

$$T_2(x) = 2x(Ax + D\sqrt{1-x^2}) - A = A(2x^2-1) + Dx\sqrt{1-x^2}$$

Where A and B are arbitrary integration constants. The first term is A times the second order Chebyshev polynomial of the first kind.
Let's proceed one step further, where we use the fact that the constants D in $T_1(x)$ and $T_2(x)$ are quite arbitrary and hence may be set to different values, say $D = D_1$ and $D = D_2$ respectively:

$$T_3(x) = 2x\left[A(2x^2-1) + D_2x\sqrt{1-x^2}\right] - \left[Ax + D_1\sqrt{1-x^2}\right]$$

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> simplify((1-x^2)*diff(diff(T3(x),x),x)-x*diff(T3(x),x)+3^2*T3(x));
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$$\frac{4(-D_2x^2 + 2D_1x^2 + D_2 - 2D_1)}{\sqrt{1-x^2}}$$

So $T_3(x)$ is a solution if and only if $D_2 = 2D_1$. We have re-discovered the Chebyshev polynomials of the second kind. The factor 2 is in the $U_2(x)$ definition below:

$$\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_{k+1}(x) &= 2x\,U_k(x) - U_{k-1}(x) \end{aligned}$$

We have a problem with the notation by now. So far, we have reserved the names T_n for denoting *all* solution functions of the Chebyshev differential equation. But it is better to restrict these names to Chebyshev polynomials of the first kind, as it is usually done:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{k+1}(x) &= 2x\,T_k(x) - T_{k-1}(x) \end{aligned}$$

And we shall introduce a brand new notation $S_n(x)$ for the Solutions of the Chebyshev differential equation. With the third step in the solution process:

$$S_3(x) = AT_3(x) + B\sqrt{1-x^2}\,U_2(x) = A(4x^3-3x) + B\sqrt{1-x^2}\,(4x^2-1)$$

Where A and B are arbitrary integration constants.
Now we wonder if this leaves us alone with Chebyshev Polynomials of the first and second kind as proper solutions of the CDE for all orders n . But let's first summarize the solutions that we already have:

$$\begin{aligned} S_0(x) &= A + B\arccos(x) \\ S_1(x) &= Ax + B\sqrt{1-x^2}\,.\,1 \\ S_2(x) &= A(2x^2-1) + B\sqrt{1-x^2}\,.\,2x \\ S_3(x) &= A(4x^3-3x) + B\sqrt{1-x^2}\,(4x^2-1) \end{aligned}$$

All other solutions can be obtained with help of the recurrence relationship $S_{n+1} = 2x\,S_n(x) - S_{n-1}(x)$, which holds for all solution functions as well as for both the polynomials of the first and second kind. So it follows by superposition and mathematical induction that the general solution of Pafnuty's differential equation for $n \geq 1$ a natural number is expressed in Chebyshev polynomials of the first $T_n(x)$ and second $U_n(x)$ kind:

$$S_n(x) = A\,T_n(x) + B\sqrt{1-x^2}\,U_{n-1}(x)$$

Together with a special solution $S_0(x) = A + B\arccos(x)$. The formula for $n \geq 1$ is confirmed by the closed form (29) on the Mathworld web page:

`http://mathworld.wolfram.com/ChebyshevDifferentialEquation.html`

Half Integer Solutions

Pafnuty Chebyshev's differential equation cannot be copied and pasted enough. It is a bit changed due to our new notation conventions, though:

$$(1-x^2)\frac{d^2S_n(x)}{dx^2} - x\frac{dS_n(x)}{dx} + n^2S_n(x) = 0$$

Apart from the special case $S_0(x) = A + B\arccos(x)$ for $n = 0$ in the differential equation, we have found two other *Special Solutions*, which are associated with half-integer values $n = 1/2$ and $n = -1/2$:

$$S_{-1/2}(x) = S_{+1/2}(x) = A\sqrt{1-x} + B\sqrt{1+x}$$

Also the *ladder operators* cannot be copied and pasted enough to remember:

$$\begin{aligned} \left[(1-x^2)\frac{d}{dx} - n\,x\right] S_n(x) &= -n\,S_{n+1}(x) \\ \left[(1-x^2)\frac{d}{dx} + n\,x\right] S_n(x) &= +n\,S_{n-1}(x) \end{aligned}$$

It turns out that we have to be careful with the two special solutions and use the second of the two ladder relations to derive the following:

$$(1-x^2)\frac{dS_{1/2}}{dx} + \frac{1}{2}\,x\,S_{1/2}(x) = \frac{1}{2}S_{-1/2}(x)$$

With $S_{1/2}(x) = \sqrt{1+x}$:

$$(1-x^2)\frac{1/2}{\sqrt{1+x}} + \frac{1}{2}\,x\sqrt{1+x} = \frac{1}{2}(1-x)\sqrt{1+x} + \frac{1}{2}\,x\sqrt{1+x} \quad \Longrightarrow$$

$$\frac{1}{2}\sqrt{1+x} = \frac{1}{2}S_{-1/2}(x) \quad \Longrightarrow \quad S_{-1/2}(x) = S_{1/2}(x) = \sqrt{1+x}$$

With $S_{1/2}(x) = \sqrt{1-x}$:

$$(1-x^2)\frac{-1/2}{\sqrt{1-x}} + \frac{1}{2}\,x\sqrt{1-x} = -\frac{1}{2}(1+x)\sqrt{1-x} + \frac{1}{2}\,x\sqrt{1-x} \quad \Longrightarrow$$

$$-\frac{1}{2}\sqrt{1-x} = \frac{1}{2}S_{-1/2}(x) \quad \Longrightarrow \quad S_{-1/2}(x) = -S_{1/2}(x) = -\sqrt{1-x}$$

The minus sign in the last formula is important. But, now we have found two basis functions to start the recursion $S_{n+1}(x) = 2x\,S_n(x) - S_{n-1}(x)$:

$$S_{3/2}(x) = 2x\,S_{1/2}(x) - S_{-1/2}(x)$$

With $S_{1/2}(x) = \sqrt{1+x}$:

$$S_{3/2}(x) = 2x\,\sqrt{1+x} - \sqrt{1+x} = (2x-1)\sqrt{1+x}$$

With $S_{1/2}(x) = \sqrt{1+x}$:

$$S_{3/2}(x) = 2x\,\sqrt{1-x} + \sqrt{1-x} = (2x+1)\sqrt{1-x}$$

The general solution is, of course, a linear combination of the two:

$$S_{3/2}(x) = A\,(2x-1)\sqrt{1+x} + B\,(2x+1)\sqrt{1-x}$$

Let's check this with MAPLE:

```
> S(x) := A*(2*x-1)*sqrt(1+x) + B*(2*x+1)*sqrt(1-x);
> simplify((1-x^2)*diff(diff(S(x),x),x)-x*diff(S(x),x)+(3/2)^2*S(x));
```

$$0$$

Proceeding in this way, we can construct all *half integer solutions* of Pafnuty Chebyshev's differential equation. They are of the form:

$$S_{(2n+1)/2}(x) = A\,P_n(x)\,\sqrt{1+x} + B\,Q_n(x)\,\sqrt{1-x}$$

Where $n \geq 0$ is a natural and the polynomials $P_n(x)$ and $Q_n(x)$ are generated by the following recursion relations:

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= 2x - 1 \\
P_{n+1}(x) &= 2x\, P_n(x) - P_{n-1}(x)
\end{aligned}
$$

$$
\begin{aligned}
Q_0(x) &= 1 \\
Q_1(x) &= 2x + 1 \\
Q_{n+1}(x) &= 2x\, Q_n(x) - Q_{n-1}(x)
\end{aligned}
$$

Derivatives

The ladder relations have become exactly as in paragraph 3.4 of the paper on *One-parameter isospectral special functions*:

$$
\begin{aligned}
\left[(1-x^2)\frac{d}{dx} - n\,x\right] T_n(x) &= -n\, T_{n+1}(x) \\
\left[(1-x^2)\frac{d}{dx} + n\,x\right] T_n(x) &= +n\, T_{n-1}(x)
\end{aligned}
$$

If we add these equations togther again, then:

$$2(1-x^2)\frac{dT_n}{dx} = -n\,[T_{n+1}(x) - T_{n-1}(x)]$$

Giving for the derivative of a Chebyshev solution function:

$$\frac{dT_n}{dx} = \frac{n}{2(x^2-1)}\,[T_{n+1}(x) - T_{n-1}(x)]$$

Remember the recursion relationship:

$$T_{n+1}(x) + T_{n-1}(x) = 2x\, T_n(x)$$

With help of this, we can do the following:

$$
\begin{gathered}
T_{n+1}(x) - T_{n-1}(x) = 2x\, T_n(x) - 2T_{n-1}(x) = \\
2x\,[2\,xT_{n-1}(x) - T_{n-2}(x)] - 2T_{n-1}(x) = \\
4(x^2-1)T_{n-1}(x) + 2T_{n-1}(x) - 2x\, T_{n-2}(x) = \\
4(x^2-1)T_{n-1}(x) + 2T_{n-1}(x) - [T_{n-1}(x) + T_{n-3}(x)] = \\
4(x^2-1)T_{n-1}(x) + [T_{n-1}(x) - T_{n-3}(x)]
\end{gathered}
$$

Herewith:

$$\frac{dT_n}{dx} = \frac{n}{2(x^2-1)}\,[T_{n+1}(x) - T_{n-1}(x)] =$$

$$2nT_{n-1}(x) + \frac{n}{2(x^2-1)}\left[T_{n-1}(x) - T_{n-3}(x)\right]$$

So we have a recursion formula expressing the derivative of Chebyshev function into a sequence of lower order Chebyshev functions:

$$\frac{dT_n}{dx} = 2nT_{n-1}(x) + 2nT_{n-3}(x) + 2nT_{n-5}(x) + \ \ldots \ + (\text{last term})$$

Where the last term can be one of these two:

$$\frac{n}{2(x^2-1)}\left[T_3(x) - T_1(x)\right] \quad \text{or} \quad \frac{n}{2(x^2-1)}\left[T_2(x) - T_0(x)\right]$$

If solutions of the Chebyshev differential equation are restricted to Chebyshev polynomials of the first kind, then we have $T_3(x) = 4x^3 - 3x$, $T_2(x) = 2x^2 - 1$, $T_1(x) = x$, $T_0(x) = 1$. Herewith:

$$\frac{n}{2(x^2-1)}\left[T_3(x) - T_1(x)\right] = 2n.x \quad \text{or} \quad \frac{n}{2(x^2-1)}\left[T_2(x) - T_0(x)\right] = n$$

Let's concentrate on the value of the derivative at $x = 1$. From the definition of the Chebyshev polynomials $T_n(x) = \cos[h](n \arccos[h](x))$ it is clear that, for all $n : T_n(1) = 1$. And $x = 1$ is the only place where *all* polynomials assume that maximum value, at the interval $[-1, +1]$. Thus the derivatives $dT_n(x)/dx$ assume their maximum value at that point as well. We can even calculate what the maximum is. For even n we have $n/2$ terms $2n$, giving a total of $n/2.2n = n^2$. For odd n we have $(n-1)/2$ terms equal to $2n$ and one term equal to n, giving a total of $(n-1)/2.2n + n = n^2$. Both sequences add up to n^2. Thus:

$$\left.\frac{dT_n(x)}{dx}\right|_{x=1} = n^2$$

Generating Functions

Employed is only the recurrence relation which is valid for all of the solutions of the Chebyshev differential equation:

$$S_{n+1}(x) + S_{n-1}(x) = 2xS_n(x)$$

Let $F(z, x)$ be the Generating Function, then by definition:

$$F(z, x) = S_0(x) + S_1(x).z + S_2(x).z^2 + S_3(x).z^3 \ + \ \ldots.$$

Hence:

$$z^2F(z, x) = S_0(x).z^2 + S_1(x).z^3 + S_2(x).z^4 + S_3(x).z^5 \ + \ \ldots.$$

Add these two equations together:

$$F(z, x) + z^2.F(z, x) = S_0(x) + S_1(x).z +$$

$$[S_2(x) + S_0(x)]\, z^2 + [S_3(x) + S_1(x)]\, z^3 \ + \ \ldots .$$

And use the recurrence relation:

$$F(z,x) + z^2.F(z,x) = S_0(x) + S_1(x).z - 2xz.S_0(x) +$$

$$2xS_0(x)z + 2xS_1(x)z^2 + 2xS_2(x)z^3 \ + \ \ldots .$$

$$= (1-2xz)S_0(x) + z.S_1(x) + 2xz\left[S_0(x) + S_1(x).z + S_2(x).z^2 \ + \ \ldots . \right]$$

$$= (1-2xz)S_0(x) + z.S_1(x) + 2xz.F(z,x)$$

Solve for $F(z,x)$:

$$F(z,x)(1-2xz+z^2) = (1-2xz)S_0(x) + z.S_1(x) \quad \Longrightarrow$$

$$F(z,x) = \frac{(1-2xz)S_0(x) + z.S_1(x)}{1-2xz+z^2}$$

We have found the following initializations for Chebyshev polynomials:

$$S_0(x) = 1 \quad \text{and} \quad S_1(x) = \begin{cases} x & \text{(first kind)} \\ 2x & \text{(second kind)} \\ 2x-1 & \text{(half integer)} \\ 2x+1 & \text{(half integer)} \end{cases}$$

Respectively resulting in the following set of Generating Functions:

$$F(z,x) = \begin{cases} \frac{1-xz}{1-2xz+z^2} \\ \frac{1}{1-2xz+z^2} \\ \frac{1-z}{1-2xz+z^2} \\ \frac{1+z}{1-2xz+z^2} \end{cases}$$

We can do some more with the general formula if we factorize the denominator:

$$z^2 - 2xz + 1 = (z-\alpha)(z-\beta) \quad \Longleftrightarrow \quad \alpha + \beta = 2x \quad \text{and} \quad \alpha\beta = 1$$

Where we find:

$$\alpha = x + \sqrt{x^2-1} \quad \text{and} \quad \beta = x - \sqrt{x^2-1}$$

Split into partial fractions:

$$F(z,x) = \frac{A}{z-\alpha} + \frac{B}{z-\beta} = \frac{A(z-\beta) + B(z-\alpha)}{(z-\alpha)(z-\beta)}$$

$$= \frac{(A+B)z - (A\beta + B\alpha)}{(z-\alpha)(z-\beta)}$$

On the other hand:

$$F(z,x) = \frac{[S_1(x) - 2xS_0(x)]\, z + S_0(x)}{z^2 - 2xz + 1}$$

$$\begin{aligned} A + B &= S_1(x) - 2xS_0(x) \\ \beta A + \alpha B &= -S_0(x) \end{aligned}$$

Two equations with two unknowns. The solution is:

$$\begin{aligned} A &= +\frac{\alpha\,[S_1(x) - 2xS_0(x)] + S_0(x)}{\alpha - \beta} = +\frac{\alpha S_1(x) + [1 - 2\alpha x]\, S_0(x)}{\alpha - \beta} \\ B &= -\frac{\beta\,[S_1(x) - 2xS_0(x)] + S_0(x)}{\alpha - \beta} = -\frac{\beta S_1(x) + [1 - 2\beta x]\, S_0(x)}{\alpha - \beta} \end{aligned}$$

This can be simplified even further:

$$1 - 2x\alpha = \alpha\beta - (\alpha+\beta)\alpha = -\alpha^2 \quad \Longrightarrow \quad A = \alpha\frac{S_1(x) - \alpha S_0(x)}{\alpha - \beta}$$

$$1 - 2x\beta = \alpha\beta - (\alpha+\beta)\beta = -\beta^2 \quad \Longrightarrow \quad B = \beta\frac{-S_1(x) + \beta S_0(x)}{\alpha - \beta}$$

We proceed as follows:

$$F(z,x) = \frac{A}{z-\alpha} + \frac{B}{z-\beta} = -\frac{A}{\alpha}\frac{1}{1 - z/\alpha} - \frac{B}{\beta}\frac{1}{1 - z/\beta}$$

$$= -\frac{A}{\alpha}\left[1 + \frac{1}{\alpha}z + \frac{1}{\alpha^2}z^2 + \frac{1}{\alpha^3}z^3 + \frac{1}{\alpha^4}z^4 \;+\; \ldots .\right]$$

$$-\frac{B}{\beta}\left[1 + \frac{1}{\beta}z + \frac{1}{\beta^2}z^2 + \frac{1}{\beta^3}z^3 + \frac{1}{\beta^4}z^4 \;+\; \ldots .\right]$$

$$= -\left[\frac{A}{\alpha} + \frac{B}{\beta}\right] - \left[\frac{A}{\alpha^2} + \frac{B}{\beta^2}\right] z - \left[\frac{A}{\alpha^3} + \frac{B}{\beta^3}\right] z^2 - \left[\frac{A}{\alpha^4} + \frac{B}{\beta^4}\right] z^3 \;+\; \ldots .$$

On the other hand we have:

$$F(z,x) = S_0(x) + S_1(x).z + S_2(x).z^2 + S_3(x).z^3 \;+\; \ldots .$$

Conclusion:

$$S_n(x) = -\left[\frac{A}{\alpha}\frac{1}{\alpha^n} + \frac{B}{\beta}\frac{1}{\beta^n}\right] = \frac{-S_1(x) + \alpha S_0(x)}{\alpha - \beta}\frac{1}{\alpha^n} + \frac{S_1(x) - \beta S_0(x)}{\alpha - \beta}\frac{1}{\beta^n}$$

In all cases known to us we have that $S_0(x) = 1$. Hence:

$$S_n(x) = \frac{-S_1(x) + x + \sqrt{x^2-1}}{2\sqrt{x^2-1}}(x - \sqrt{x^2-1})^n$$

$$+\frac{S_1(x)-x+\sqrt{x^2-1}}{2\sqrt{x^2-1}}(x+\sqrt{x^2-1})^n$$

The most well-known result is the one for Chebyshev polynomials of the first kind. In that case $S_1(x) = x$ and $S_n(x) = T_n(x)$. Hence:

$$T_n(x) = \frac{(x-\sqrt{x^2-1})^n + (x+\sqrt{x^2-1})^n}{2}$$

Which appears in many other forms in the litterature and on the Internet, for example:

`http://www.focusonmath.org/FOM/resources/publications/MT2004-08-20a.pdf`
`http://en.wikipedia.org/wiki/Chebyshev_polynomial`

The only new thing here may be that such formulas can also be derived for the polynomials in the half integer solutions of the Chebyshev differential equation.

Cosine Space

Many things with Chebyshev Polynomials become more transparent if we keep in mind that there exists kind of a mapping between *Polynomial Space* and *Cosine Space*. What do we mean by this? The following.

$$T_n(x) = \cos(n \arccos(x)) \qquad \text{where} \quad -1 \le x \le +1$$

Or, in a more instructive form:

$$T_n(\theta) = \cos(n\,\theta) \quad \text{where:} \quad x = \cos(\theta) \quad \text{and} \quad -\pi \le \theta \le 0$$

Here T_n as a function of x is called *Polynomial Space* and T_n as a function of θ is called *Cosine Space*. The mapping between the two can be visualized as has been done on my web site:

`http://hdebruijn.soo.dto.tudelft.nl/jaar2006/pafnuty1.htm`
`http://hdebruijn.soo.dto.tudelft.nl/jaar2006/pafnuty2.htm`

The density $D_\theta(x)$ of the angles θ in Polynomial Space can be determined with my theory of *Exact Densities* .

$$D_\theta(x)\,dx = d\arccos(x) \quad \Longrightarrow \quad D(x) = \frac{1}{\sqrt{1-x^2}}$$

Which shows that the angles become infinitely dense near the boundaries $x = \pm 1$. The reverse problem is the density $D_x(\theta)$ of the x-coordinates in Cosine Space:

$$D_x(\theta)\,d\theta = d\,\cos(\theta) \quad \Longrightarrow \quad D_x(\theta) = -\sin(\theta)$$

Which shows that the x-coordinates have zero density near the boundaries $\theta = \{-\pi, 0\}$. So far so good. Let's go for the derivatives.

$$T_n'(\theta) = \frac{dT_n(x)}{dx}\frac{dx}{d\theta} = -T_n'(x)\sin(\theta) = -T_n'(x)\sqrt{1-x^2}$$

$$T_n''(\theta) = \frac{d}{d\theta}\left[-T_n'(x)\sin(\theta)\right] = -T_n''(x)\frac{dx}{d\theta}\sin(\theta) - T_n'(x)\cos(\theta) =$$
$$(1-x^2)T_n''(x) - x\,T_n'(x)$$

But wait! The differential equation named after Pafnuty Chebyshev is:

$$(1-x^2)\frac{d^2T_n(x)}{dx^2} - x\frac{dT_n(x)}{dx} + n^2T_n(x) = 0$$

Thus in Cosine Space, it says:

$$\frac{d^2T_n(\theta)}{d\theta^2} + n^2T_n(\theta) = 0$$

So it's no surprise anymore that its solutions are also given by $T_n(\theta) = \cos(n\,\theta)$, hence that $T_n(x) = \cos(n\arccos(x))$, which has been a mystery, until now. But it seems that we are not finished. There is also a domain of interest with this differential equation for $x > 1$. Instead of $x = \cos(\theta)$, substitute $x = \cosh(p)$. And let's go for the derivatives again:

$$T_n'(p) = \frac{dT_n(x)}{dx}\frac{dx}{dp} = T_n'(x)\sinh(p) = T_n'(x)\sqrt{\cosh^2(p)-1} = T_n'(x)\sqrt{x^2-1}$$

$$T_n''(p) = \frac{d}{dp}\left[T_n'(x)\sinh(p)\right] = T_n''(x)\frac{dx}{dp}\sinh(p) + T_n'(x)\cosh(p) =$$
$$(x^2-1)T_n''(x) + x\,T_n'(x) = -\left[(1-x^2)T_n''(x) - x\,T_n'(x)\right]$$

Thus in Hyperbolic Cosine Space, it says:

$$\frac{d^2T_n(p)}{dp^2} - n^2T_n(p) = 0$$

So it's no surprise anymore that its solutions are also given by $T_n(p) = \cosh(n\,p)$, hence that $T_n(x) = \cosh(n\,\text{arccosh}(x))$.

Final roundup. Considerable effort was spent in proving the following theorem:

$$\left.\frac{dT_n(x)}{dx}\right|_{x=1} = n^2$$

Within Cosine Space, gathering evidence is much easier:

$$T_n'(x) = \frac{dT_n(\theta)}{d\theta}\frac{d\theta}{dx} = \frac{n\,\sin(n\,\theta)}{\sin(\theta)}$$

Hence for $(x=1) \Longleftrightarrow (\theta = 0)$:

$$T_n'(1) = \lim_{\theta\to 0} n^2\frac{\sin(n\,\theta)}{n\,\theta}\frac{\theta}{\sin(\theta)} = n^2$$

Disclaimers:

Anything free comes without referee :-(
My English may be better than your Dutch.