Renormalization of Singularities

Author: Han de Bruijn
Date: 2006 June

Measuring a function in physics can be modelled by a convolution integral of this function with e.g. a Gaussian distribution. Since basically everything in physics is subject to measurement, the very *nature* of any function in mathematics is revealed by such a convolution integral. The aim of this paper is to demonstrate that several singularities of the form $1/r^d$, where d is smaller than the space dimension, are actually non-existent in nature.

Fuzzy Elektron

In general, the energy density in the electric field of a point charge q is given by $w = \frac{1}{2}\epsilon_0 E^2$ where the electric field E at a distance r is:

$$E = \frac{q}{4\pi\epsilon_0 r^2}$$

The total energy in the field is thus given by the integral:

$$U = \iiint \frac{1}{2}\epsilon_0 E^2 \, dx \, dy \, dz = \int_0^\infty \frac{1}{2}\epsilon_0 \left(\frac{q}{4\pi\epsilon_0 r^2} \right)^2 4\pi r^2 dr \ =$$

$$\frac{q^2}{8\pi\epsilon_0} \int_0^\infty \frac{dr}{r^2} = \frac{q^2}{8\pi\epsilon_0} \left[\frac{1}{r} \right]_0^\infty = \infty$$

There is an infinite outcome for de self energy of the electron. This is quite a serious problem in electrodynamics. Because of the equivalence of mass and energy (via $E = m_0 c^2$) it would mean, for example, that an electron can not move in space, at all, theoretically. We shall see now how this problem can be resolved by renormalization, as understood by this author.
The Gaussian broadening operator for one dimension is:

$$e^{\frac{1}{2}\sigma^2 (d/dx)^2}$$

We could have called it a *sensor*. The *spread* of the sensor is σ. Generalisation of the sensor S to three dimensions is straightforward:

$$S = e^{\frac{1}{2}\sigma^2 \nabla^2}$$

Where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.
If S works on a function f, then we write: $\overline{f} = S\,f$, where $\overline{f}$ may be called the *sense* of f. Provided that σ is small when compared with the overall size of the problem. We could also say, more "mathematically", that $\sigma \to 0$.
We have seen that renormalizing an *integrand* is not of much help with the prevention of divergent integrals. Because there is a theorem that the integral

of the fuzzyfication is equal to the integral of the original. It has no sense, therefore, to apply broadening on $1/r^2$, as has happened in my (Dutch) book, quite unfortunately. Thus we must do the renormalization at some other place in the derivation. *This* place, to be precise:

$$\frac{q^2}{8\pi\epsilon_0}\left[\frac{1}{r}\right]_0^\infty$$

As follows: $\overline{1/r}(x,y,z) =$

$$\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^3 \iiint \frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}}\, e^{-[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]/2\sigma^2}\; d\xi\, d\eta\, d\zeta$$

However, the Gaussian broadening is only needed at two distinct places: $r=0$, $r=\infty$. The value at $r=\infty$ is worked out with help of the well-known fact that $\overline{f} = f + \frac{1}{2}\sigma^2\nabla^2 f$: see the "Far away Field" subsection below.
When applied to the electric field around the electron (in spherical symmetric coordinates), the second order term disappears:

$$\nabla^2\frac{1}{r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}\, r\,\frac{1}{r} = 0 \quad\Longrightarrow\quad \lim_{r\to 0}\overline{1/r} = \lim_{r\to 0} 1/r\left(1+O(\left[\frac{\sigma}{r}\right]^n)\right) = 0$$

Not unexpectedly, because *any* potential function *obeys* the Laplace equation. Thus $\nabla^2 1/r = 0$ for $r>0$. Meaning that any difference with the classical solution $1/r$ will be at least of third order (i.e. order $(\sigma/r)^n$ where $n\geq 3$).
On the other hand, $r=0$ just means that $x=y=z=0$. Thus the three-dimensional convolution integral is simplified a great deal at that place. If we then substitute $\xi^2+\eta^2+\zeta^2=r^2$, it becomes:

$$\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^3\int_0^\infty \frac{1}{r}\, e^{-r^2/2\sigma^2}\, 4\pi r^2 dr =$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}(r/\sigma)^2}\, d(r/\sigma)^2 = 2\frac{1}{\sigma\sqrt{2\pi}}$$

Thus the overall result for the self energy of an electron is a finite outcome:

$$U = \frac{q^2}{4\pi\epsilon_0}\frac{1}{\sigma\sqrt{2\pi}} \approx 0.40\, V(\sigma).q$$

With other words: the self energy of an electron is a factor times the energy of the electric field of a test charge q at a distance σ.
Instead of the Gaussian shape function, let's utilize a function with a small solid sphere - radius R - as its domain of non-zero'ness and with height one divided by the sphere's volume. We can take for granted that the far-away term for $r=\infty$ is zero again. The convolution with the sphere function at the origin $r=0$ is given by:

$$\int_0^R \frac{1}{r}\frac{1}{4/3\,\pi R^3}4\pi r^2\, dr = \frac{1/2\; 4\pi R^2}{1/3\; 4\pi R^3} = \frac{3}{2}\frac{1}{R}$$

Where the spread σ is related to the well known moment of inertia of a sphere: $\sigma^2 = 1/5\, R^2 \quad \Longrightarrow \quad R = \sqrt{5}\,\sigma$. Conclusion:

$$U = \frac{q^2}{4\pi\epsilon_0} \frac{3}{4\sqrt{5}\,\sigma} \approx 0.34\, V(\sigma).q$$

It is seen that, in general, the self-energy is given by a dimensionless *form-factor*, times the energy of the electric field of a test charge q at a distance σ.

Far away Field

The result of an observation of a function $g(x, y, z)$ with the sensor S is in general a convolution integral:

$$\overline{g} = \iiint f(\xi, \eta, \zeta) g(x - \xi, y - \eta, z - \zeta) d\xi d\eta d\zeta$$

Here f is the kernel function of the sensor and g is the function to be observed. The function g is developed into a Taylor series around $(\xi, \eta, \zeta) = (0, 0, 0)$:

$$\begin{aligned} g(x - \xi, y - \eta, z - \zeta) \;\approx\; & g(x, y, z) \\ & -\xi \frac{\partial g}{\partial x} - \eta \frac{\partial g}{\partial y} - \zeta \frac{\partial g}{\partial z} \\ & + \frac{1}{2}\xi^2 \frac{\partial^2 g}{\partial x^2} + \frac{1}{2}\eta^2 \frac{\partial^2 g}{\partial y^2} + \frac{1}{2}\zeta^2 \frac{\partial^2 g}{\partial z^2} \\ & + \xi\eta \frac{\partial^2 g}{\partial x \partial y} + \eta\zeta \frac{\partial^2 g}{\partial y \partial z} + \zeta\xi \frac{\partial^2 g}{\partial z \partial x} \end{aligned}$$

In this expression the partial differential quotients are no longer dependent on (ξ, η, ζ), because they are calculated for $(\xi, \eta, \zeta) = (0, 0, 0)$. Therefore the convolution integral is approximately equal to:

$$g(x, y, z) \iiint f \, d\xi d\eta d\zeta$$

$$- \frac{\partial g}{\partial x} \iiint \xi \, f \, d\xi d\eta d\zeta \; - \; \frac{\partial g}{\partial y} \iiint \eta \, f \, d\xi d\eta d\zeta \; - \; \frac{\partial g}{\partial z} \iiint \zeta \, f \, d\xi d\eta d\zeta \; +$$

$$\frac{1}{2} \frac{\partial^2 g}{\partial x^2} \iiint \xi^2 \, f \, d\xi d\eta d\zeta \; + \; \frac{1}{2} \frac{\partial^2 g}{\partial y^2} \iiint \eta^2 \, f \, d\xi d\eta d\zeta \; + \; \frac{1}{2} \frac{\partial^2 g}{\partial z^2} \iiint \zeta^2 \, f \, d\xi d\eta d\zeta +$$

$$\frac{\partial^2 g}{\partial x \partial y} \iiint \xi\eta \, f \, d\xi d\eta d\zeta \; + \; \frac{\partial^2 g}{\partial y \partial z} \iiint \eta\zeta \, f \, d\xi d\eta d\zeta \; + \; \frac{\partial^2 g}{\partial z \partial x} \iiint \zeta\xi \, f \, d\xi d\eta d\zeta$$

The first integral is by definition equal to 1. The next three integrals are equal to the expectation value of the kernel function, and therefore equal to 0. The next three integrals are equal to the spreads of the kernel function in the different coordinate directions. The last three integrals, at last, are zero. Thus:

$$\overline{g} \approx g + \frac{1}{2}\sigma_x^2 \frac{\partial^2 g}{\partial x^2} + \frac{1}{2}\sigma_y^2 \frac{\partial^2 g}{\partial y^2} + \frac{1}{2}\sigma_z^2 \frac{\partial^2 g}{\partial z^2} = g + \frac{1}{2}\sigma^2 \nabla^2 g$$

An asymptotic approximation for large distances, which can also be found in any decent book about Statistics, where the last equality is for the isotropic case only (which in turn is guaranteed by spherical symmetry).

Sphere Singularity

We repeat, the electric field strength of a point charge q at a distance r is:

$$E = \frac{q}{4\pi\epsilon_0 r^2}$$

Instead of renormalizing the total energy in the field, we can try on the electric field itself. As follows: $\overline{1/r^2}(x, y, z) =$

$$\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^3 \iiint \frac{1}{\xi^2 + \eta^2 + \zeta^2}\, e^{-[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]/2\sigma^2}\, d\xi\, d\eta\, d\zeta$$

Values at $r = \infty$ (r large) are worked out with help of the well-known fact that $\overline{f} = f + \frac{1}{2}\sigma^2\nabla^2 f$: see the "Far away Field" subsection above.
When applied to the electric field around the electron (in spherical symmetric coordinates), the second order term now becomes:

$$\nabla^2 \frac{1}{r^2} = \frac{1}{r}\frac{\partial^2}{\partial r^2} r \frac{1}{r^2} = \frac{2}{r^4} \quad \Longrightarrow \quad \overline{E} \approx E\left(1 + \frac{\sigma^2}{r^2}\right)$$

Meaning that the electric field, even if it's only moderately far away from the singularity, can hardly be distinguished, if at all, from the classical field. Thus the Gaussian broadening is essentially felt at the singularity only: $r = 0$. On the other hand, $r = 0$ just means that $x = y = z = 0$. Thus the three-dimensional convolution integral is simplified a great deal at that place. If we then substitute $\xi^2 + \eta^2 + \zeta^2 = r^2$, it becomes:

$$\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^3 \int_0^\infty \frac{1}{r^2}\, e^{-r^2/2\sigma^2} 4\pi r^2 dr = \frac{4\pi\, 1/2\, \sigma\sqrt{2\pi}}{\sigma^2 2\pi\, \sigma\sqrt{2\pi}} = \frac{1}{\sigma^2}$$

$$\Longrightarrow \quad \overline{E}(0) = \frac{q}{4\pi\epsilon_0\sigma^2}$$

A very beautiful result! Now let's do the same thing with the sphere function of the preceding paragraph. And employ the fact that $R = \sqrt{5}\,\sigma$:

$$\int_0^R \frac{1}{r^2}\frac{1}{4/3\,\pi R^3} 4\pi r^2\, dr = \frac{4\pi R}{1/3\, 4\pi R^3} = \frac{3}{R^2} = \frac{3}{5}\frac{1}{\sigma^2}$$

$$\Longrightarrow \quad \overline{E}(0) = 0.6\, \frac{q}{4\pi\epsilon_0\sigma^2}$$

It's seen, again, how the *form-factor* of the shape functions come into play. Again, the form factor of the Gaussian is the greatest of the two.

Cylinder Singularity

Instead of three-dimensional singularities $1/r$ and $1/r^2$, consider a singularity $1/r$ in two dimensions only. Together with Gaussian and with "rectangular" smoothing. The accompanying convolution integral is simplified a great deal at the singularity itself, for $r = 0$:

$$\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^2 \int_0^\infty \frac{1}{r}\, e^{-r^2/2\sigma^2} 2\pi r dr = \frac{2\pi\, 1/2\, \sigma\sqrt{2\pi}}{\sigma^2\, 2\pi} = \frac{\sqrt{\pi/2}}{\sigma}$$

Let's do the same thing with a cylinder function, which is zero outside the circle $x^2 + y^2 = R^2$ and has height $1/(\pi R^2)$ inside the same circle, resulting in a volume normed to one $= 1$. Use the fact that $\sigma = R/2$ for a cylinder. Then:

$$\int_0^R \frac{1}{r}\, \frac{1}{\pi R^2}\, 2\pi r\, dr = \frac{2\pi R}{\pi R^2} = \frac{2}{R} = \frac{1}{\sigma}$$

It's seen, again, how the *form-factor* of the shape functions come into play. Again, the form factor of the Gaussian is the greatest of the two.
Question: is the form factor of the Gaussian always the greatest of all possible form factors, for all singularities?

Useful Lemmas

In order to arrive at a suitable approximation of some integrals later on, a couple of lemmas are useful.
Lemma.

$$\int_0^\infty e^{-x^2/2} x^n dx = (n-1) \int_0^\infty e^{-x^2/2} x^{n-2} dx$$

Proof.

$$\int_0^\infty e^{-x^2/2} x^n dx = \int_0^\infty e^{-x^2/2}\, d\frac{x^{n+1}}{n+1} = \left[e^{-x^2/2} \frac{x^{n+1}}{n+1} \right]_0^\infty - \int_0^\infty \frac{x^{n+1}}{n+1} de^{-x^2/2}$$

$$= 0 - \int_0^\infty \left[-\frac{x^{n+1}}{n+1} x \right] e^{-x^2/2} dx = \int_0^\infty \frac{x^{n+2}}{n+1} e^{-x^2/2} dx \quad \Longrightarrow$$

$$(n+1) \int_0^\infty e^{-x^2/2} x^n dx = \int_0^\infty x^{n+2} e^{-x^2/2} dx$$

Now replace n by $n - 2$, read the formula from the right to the left and we are done. But the other way around is easier:

$$\int_0^\infty e^{-x^2/2} x^n dx = -\int_0^\infty x^{n-1} d\, e^{-x^2/2} = -\left[e^{-x^2/2} x^{n-1} \right]_0^\infty + \int_0^\infty e^{-x^2/2} d\, x^{n-1}$$

$$\Longrightarrow \quad \int_0^\infty e^{-x^2/2} x^n dx = (n-1) \int_0^\infty e^{-x^2/2} x^{n-2} dx$$

Let $F(n) = \int_0^\infty \exp(-x^2/2)x^n dx$ in the sequel.
Two cases can be distinguished: n is even and n is odd. In case n is even, recursion with respect to n proceeds as follows: $F(n) =$

$$(n-1)F(n-2) = (n-1)(n-3)F(n-4) = ... = (n-1)(n-3)(n-5)...1F(0)$$

Where:

$$F(0) = \int_0^\infty e^{-x^2/2} dx = \frac{1}{2}\sqrt{2\pi}$$

Substitute $n = 2m$. Then:

$$(n-1)(n-3)(n-5)...3.1 = (2m-1)(2m-3)(2m-5)...3.1 =$$

$$\frac{2m(2m-1)(2m-2)(2m-3)...3.2.1}{2m(2m-2)(2m-4)...4.2} = \frac{(2m)!}{2^m\, m(m-1)(m-2)...2.1} = \frac{(2m)!}{2^m\, m!}$$

In case n is odd, recursion with respect to n proceeds as follows: $F(n) =$

$$(n-1)F(n-2) = (n-1)(n-3)F(n-4) = ... = (n-1)(n-3)(n-5)...2F(1)$$

Where:

$$F(1) = \int_0^\infty e^{-x^2/2} x dx = \int_0^\infty e^{-x^2/2} d(x^2/2) = -\left[e^{-t}\right]_0^\infty = -(0-1) = 1$$

Substitute $n = 2m + 1$. Then:

$$(n-1)(n-3)(n-5)...3.1 = 2m(2m-2)(2m-4)...2 = 2^m\, m!$$

Most of the time, though, the integrals to be evaluated look like this:

$$\int_0^\infty e^{-\frac{1}{2}(\rho/\sigma)^2} \rho^n\, d\rho = \sigma^n \sigma \int_0^\infty e^{-\frac{1}{2}(\rho/\sigma)^2} \left(\frac{\rho}{\sigma}\right)^n d\left(\frac{\rho}{\sigma}\right) = F(n)\, \sigma^{n+1}$$

Summarizing (m is a natural number $m \geq 0$):

$$\int_0^\infty e^{-\frac{1}{2}(\rho/\sigma)^2} \rho^{2m}\, d\rho = \frac{(2m)!}{2^m\, m!}\, \sigma^{2m+1} \frac{1}{2}\sqrt{2\pi}$$

$$\int_0^\infty e^{-\frac{1}{2}(\rho/\sigma)^2} \rho^{2m+1}\, d\rho = 2^m\, m!\, \sigma^{2m+1} \qquad \text{(not applied)}$$

Lemma.

$$\int_0^{2\pi} \cos^n(x)\, dx = \frac{n-1}{n} \int_0^{2\pi} \cos^{n-2}(x)\, dx$$

Proof.

$$\int_0^{2\pi} \cos^n(x)\, dx = \int_0^{2\pi} \cos^{n-1}(x)\, d\sin(x) =$$

$$\left[\cos^{n-1}(x)\sin(x)\right]_0^{2\pi} - \int_0^{2\pi} \sin(x)\, d\cos^{n-1}(x) =$$

$$0+\int_0^{2\pi}(n-1)\cos^{n-2}(x)\sin^2(x)\,dx=$$

$$(n-1)\int_0^{2\pi}\cos^{n-2}(x)\,dx-(n-1)\int_0^{2\pi}\cos^{n}(x)\,dx \quad\Longrightarrow$$

$$\int_0^{2\pi}\cos^{n}(x)\,dx+(n-1)\int_0^{2\pi}\cos^{n}(x)\,dx=(n-1)\int_0^{2\pi}\cos^{n-2}(x)\,dx$$

Which leads to the desired result. Let $F(n)=\int_0^{2\pi}\cos^n(x)dx$ in the sequel. Two cases can be distinguished: n is even and n is odd. In case n is even, recursion with respect to n proceeds as follows:

$$F(n)=\frac{n-1}{n}F(n-2)=\frac{n-1}{n}\frac{n-3}{n-2}F(n-4)=...=\frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{1}{2}F(0)$$

Where $F(0)=\int_0^{2\pi}dx=2\pi$. In case n is odd, recursion with respect to n proceeds as follows:

$$F(n)=\frac{n-1}{n}F(n-2)=\frac{n-1}{n}\frac{n-3}{n-2}F(n-4)=...=\frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{2}{3}F(1)$$

Where $F(1)=\int_0^{2\pi}\cos(x)\,dx=0$. Thus all odd functions F are zero. The case $n=2m$ is even can be simplified further:

$$\frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{1}{2}=\frac{n(n-1)(n-2)(n-3)\ldots 3.2.1}{\left[(n(n-2)(n-4)\ldots 4.2\right]^2}=\frac{(2m)!}{(2^m\,m!)^2}\quad\Longrightarrow$$

$$\int_0^{2\pi}\cos^{2m}(x)dx=\frac{(2m)!}{(2^m\,m!)^2}\,2\pi$$

Whole Cylinder Field

The whole renormalized cylinder field is given by:

$$\overline{1/r}(x,y)=\iint\frac{e^{-\frac{1}{2}\left[(x-\xi)^2+(y-\eta)^2\right]/\sigma^2}}{2\pi\sigma^2\sqrt{\xi^2+\eta^2}}\,d\xi\,d\eta$$

We seek for a simplification of this expression. Transform to polar coordinates by $\xi=\rho\cos(\theta)$, $\eta=\rho\sin(\theta)$ and $x=r\cos(\phi)$, $y=r\sin(\phi)$:

$$\overline{1/r}(x,y)=\iint\frac{e^{-\frac{1}{2}\left[x^2-2x\xi+\xi^2+y^2-2y\eta+\eta^2\right]/\sigma^2}}{2\pi\sigma^2\rho}\,d\xi\,d\eta=$$

$$\frac{e^{-\frac{1}{2}(x^2+y^2)/\sigma^2}}{2\pi\sigma^2}\iint\frac{1}{\rho}e^{-\frac{1}{2}(\xi^2+\eta^2)/\sigma^2}e^{(x\xi+y\eta)/\sigma^2}d\xi\,d\eta$$

Where $x\xi + y\eta = r\cos(\phi)\rho\cos(\theta) + r\sin(\phi)\rho\sin(\theta) = r\rho\cos(\theta - \phi)$. Hence:

$$\overline{1/r}(x,y) = \frac{e^{-\frac{1}{2}(r/\sigma)^2}}{2\pi\sigma^2} \int_{\rho=0}^{\rho=\infty} \int_{\theta=0}^{\theta=2\pi} e^{-\frac{1}{2}(\rho/\sigma)^2} e^{r\rho\cos(\theta-\phi)/\sigma^2} \frac{1}{\rho}\rho \, d\rho \, d\theta$$

The singularity disappears by $\rho/\rho = 1$. The integral between square brackets [ ] is an integral over a periodic function with a period 2π equal to the integration interval. Consequently it is independent of ϕ. Thus the final expression to be calculated is:

$$\overline{1/r}(x,y) = \frac{e^{-\frac{1}{2}(r/\sigma)^2}}{2\pi\sigma^2} \int_0^\infty e^{-\frac{1}{2}(\rho/\sigma)^2} \left[\int_0^{2\pi} e^{r\rho\cos(\theta)/\sigma^2} d\theta \right] d\rho$$

The function $\exp(r\rho\cos(\theta)/\sigma^2)$ is expanded into a power series:

$$e^{r\rho\cos(\theta)/\sigma^2} = 1 + r\rho\cos(\theta)/\sigma^2 + \frac{\left[r\rho\cos(\theta)/\sigma^2\right]^2}{2} + \frac{\left[r\rho\cos(\theta)/\sigma^2\right]^3}{3!} +$$

$$... + \frac{\left[r\rho\cos(\theta)/\sigma^2\right]^n}{n!} + ...$$

If we integrate this general term over $\theta = 0 ... 2\pi$, then, according to the above lemmas, the result is zero for n is odd. And for $n = 2m$ is even it becomes:

$$... + \frac{\left[r\rho/\sigma^2\right]^{2m}}{(2m)!} \frac{(2m)!}{(2^m\, m!)^2} 2\pi + ...$$

Thus the integrals over ρ are of the form:

$$... + \frac{\left[r/\sigma^2\right]^{2m}}{(2^m\, m!)^2} 2\pi \int_0^\infty e^{-\frac{1}{2}(\rho/\sigma)^2} \rho^{2m} \, d\rho + ...$$

Where, according to the lemmas:

$$\int_0^\infty e^{-\frac{1}{2}(\rho/\sigma)^2} \rho^{2m} \, d\rho = \frac{1}{2}\sqrt{2\pi} \, \frac{(2m)!}{2^m\, m!} \, \sigma^{n+1}$$

Collecting terms:

$$\frac{e^{-\frac{1}{2}(r/\sigma)^2}}{2\pi\sigma^2} \, \frac{\left[r/\sigma^2\right]^{2m}}{(2^m\, m!)^2} \, 2\pi \, \frac{1}{2}\sqrt{2\pi} \, \frac{(2m)!}{2^m\, m!} \, \sigma^{n+1}$$

Hence the m-th term in the series expansion becomes:

$$e^{-\frac{1}{2}(r/\sigma)^2} \, \frac{\frac{1}{2}\sqrt{2\pi}}{\sigma} \, \frac{1}{m!} \frac{(2m)!}{4^m\, m!\, m!} \, \left[\frac{1}{2} \left(\frac{r}{\sigma} \right)^2 \right]^m$$

Where it is noted that $2m(2m-1) < 4m$. Therefore:

$$\frac{(2m)!}{4^m\, m!\, m!} < \frac{4^m\, m!\, m!}{4^m\, m!\, m!} = 1$$

Which makes that the m-th term in the series expansion is less than:

$$e^{-\frac{1}{2}(r/\sigma)^2} \; \frac{\frac{1}{2}\sqrt{2\pi}}{\sigma} \; \frac{1}{m!} \; \left[\frac{1}{2}\left(\frac{r}{\sigma}\right)^2\right]^m$$

Summing up to:

$$e^{-\frac{1}{2}(r/\sigma)^2} \; \frac{\frac{1}{2}\sqrt{2\pi}}{\sigma} \; e^{+\frac{1}{2}(r/\sigma)^2} = \frac{\frac{1}{2}\sqrt{2\pi}}{\sigma}$$

It is concluded that the series must converge and that the end-result will be smaller than $\frac{1}{2}\sqrt{2\pi}/\sigma$. The end-result is:

$$\overline{1/r}(r) = e^{-\frac{1}{2}(r/\sigma)^2} \; \frac{\frac{1}{2}\sqrt{2\pi}}{\sigma} \; \left\{ 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{(2m)!}{4^m \, m! \, m!} \; \left[\frac{1}{2}\left(\frac{r}{\sigma}\right)^2\right]^m \right\}$$

Whole Sperical Field

The whole renormalized sperical field is given by:

$$\overline{1/r^2}(x,y,z) = \iiint \frac{e^{-\frac{1}{2}\left[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\right]/\sigma^2}}{(\sigma\sqrt{2\pi})^{3/2}(\xi^2+\eta^2+\zeta^2)} \, d\xi \, d\eta \, d\zeta$$

We seek for a simplification of this expression. Transform to spherical coordinates by $\xi = \rho\,\cos(\phi)\sin(\theta)$, $\eta = \rho\,\sin(\phi)\sin(\theta)$ and $\zeta = \rho\,\cos(\theta)$. Since the whole problem is sperically symmetric, it's sufficient to calculate the field in the direction of the z-axis only: $x = 0$, $y = 0$, $z = r$:

$$\overline{1/r^2}(r) = \iiint \frac{e^{-\frac{1}{2}\left[(\rho\sin(\theta))^2+(r-\rho\cos(\theta))^2\right]/\sigma^2}}{(\sigma\sqrt{2\pi})^{3/2}\rho^2} \, d\rho \, \rho\sin(\theta) d\phi \, \rho d\theta =$$

$$\frac{e^{-\frac{1}{2}(r/\sigma)^2}}{(\sigma\sqrt{2\pi})^{3/2}} \int_0^{\infty} e^{-\frac{1}{2}(\rho/\sigma)^2} \left[\int_0^{\pi} e^{(r\rho\cos(\theta))/\sigma^2} \sin(\theta)\, d\theta\right] d\rho \; \int_0^{2\pi} d\phi$$

Where $\int_0^{2\pi} d\phi = 2\pi$. The singularity disappears by $\rho^2/\rho^2 = 1$. The integral between square brackets [ ] is:

$$-\int_0^{\pi} e^{(r\rho\cos(\theta))/\sigma^2} \, d\cos(\theta) = \int_{-1}^{+1} e^{r\rho t/\sigma^2} \, dt = \frac{e^{+r\rho/\sigma^2} - e^{-r\rho/\sigma^2}}{r\rho/\sigma^2}$$

And the final expression to be calculated is:

$$\overline{1/r^2}(r) = \frac{e^{-\frac{1}{2}(r/\sigma)^2}}{\sigma^2} \left[\frac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(\rho/\sigma)^2}}{\sigma\sqrt{2\pi}} \; \frac{e^{+r\rho/\sigma^2} - e^{-r\rho/\sigma^2}}{r\rho/\sigma^2} \, d\rho\right]$$

A function $\sinh(x)/x = (\exp(+x) - \exp(-x))/(2x)$ is recognized and expanded into a power series:

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \,... - 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} \,...\right)/(2x)$$

$$1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \, ... + \frac{x^{2m}}{(2m+1)!}$$

Herewith the expression between square brackets [ ] becomes:

$$\int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(\rho/\sigma)^2}}{\sigma\sqrt{2\pi}} \left[1 + \sum_{m=1}^{\infty} \frac{(r\rho/\sigma^2)^{2m}}{(2m+1)!} \right] d\rho$$

Exchanging summation and integration gives rise to terms:

$$\frac{(r/\sigma)^{2m}}{(2m+1)!} \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(\rho/\sigma)^2}}{\sqrt{2\pi}} (\rho/\sigma)^{2m} \, d(\rho/\sigma) =$$

According to the Useful Lemmas:

$$= \frac{(r/\sigma)^{2m}}{(2m+1).(2m)!} \frac{(2m)!}{2^m \, m!} = \frac{\left[\frac{1}{2}(r/\sigma)^2\right]^m}{(2m+1)\, m!}$$

Thus the end-result is:

$$\overline{1/r^2}(r) = \frac{e^{-\frac{1}{2}(r/\sigma)^2}}{\sigma^2} \left[1 + \sum_{m=1}^{\infty} \frac{\left[\frac{1}{2}(r/\sigma)^2\right]^m}{(2m+1)\, m!} \right]$$

Without the factor $(2m+1)$ in the denominator, the outcome would be:

$$\frac{e^{-\frac{1}{2}(r/\sigma)^2}}{\sigma^2} e^{+\frac{1}{2}(r/\sigma)^2} = \frac{1}{\sigma^2}$$

Thus the series converges and the outcome is less than $1/\sigma^2$ everywhere.

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch.