Fuzzy Analysis

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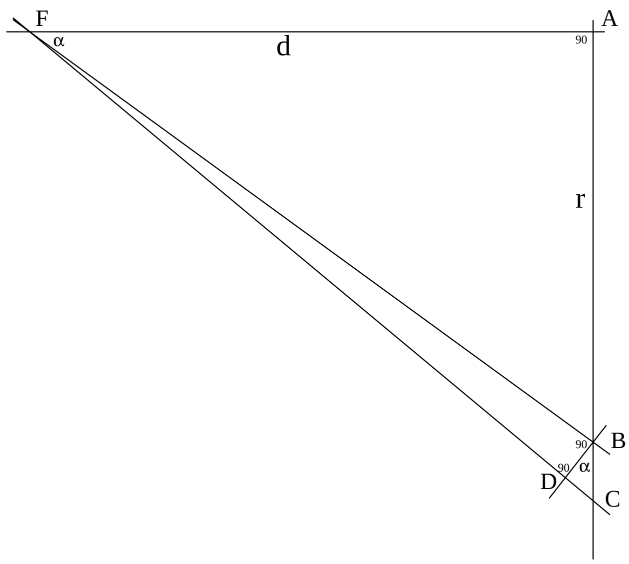
Date: 2006 March

Fuzzy Optics

Sometimes you can see things better with your eyes half shut. Maybe there is no more lucid way than this for expressing the idea of *continuity*. In the physics lessons at school, a little piece of geometric optics has always been part of the program: convex and concave mirrors and lenses. Herewith it is assumed, quite naturally, that any image of an object is *crisp* and clear. My proposal here is to say goodbye to this good habit, and pay attention to *fuzzy* images instead. In the figure below, much enlarged, we see the geometry of such a fuzzy image:



Usually, a crisp image is formed at the spot F. However, now suppose that the image plane is shifted a little bit to the right, over a distance d . Consider a very narrow light bundle FBC. The bundle fans out slowly and hits the image plane at BC. Because the bundle is very narrow, both the angles FBD and FDB are approximately 90 degrees. This means that the angle CBD will be approximately equal to the angle BFA. Name this angle α. The light density

P at the surface BC shall be calculated. Assume that the light is emitted by a point source with strength 1, then: $P = cos(\alpha)/2\pi R^2$ (: half sphere). Here $cos(\alpha) = d/R$ and $R = \sqrt{r^2 + d^2}$. If the surface BC is contracted to a point, then we find for the light strength in place the following "exact" expression:

$$P(r) = \frac{d/2\pi}{(r^2 + d^2)^{3/2}}$$

It is noted that the derivation with help of the approximately straight angles FBD and FDB is motivated only afterwards, as the limit "has been taken". This is a typical example of a "derivation with pain", as it is applied quite frequently in the applied sciences.
A few things are noted. At first that a crisp image is obtained (a *delta function* to be precise) as soon as the distance d approaches zero. Integration of the formula over the whole image plane obviously must yield a total amount of light equal to 1 . We can check out this:

$$\iint \frac{d/2\pi}{(r^2 + d^2)^{3/2}}\, r.dr.d\phi = 2\pi.\frac{1}{2\pi}.\int_0^\infty \frac{r/d.d(r/d)}{\left[(r/d)^2 + 1\right]^{3/2}} = -\frac{2}{2}.\left[x^{-1/2}\right]_1^\infty = 1$$

If a straight line is considered, instead of a point-like light source, then the function P must be integrated all over this line. Along the line, measurement is defined with a length l. The radius r in the above formulas is replaced by $p^2 + l^2$, where p is the distance of the point (x, y) to the line. The integration procedure therefore is as follows:

$$L = \int_{-\infty}^{+\infty} \frac{d/2\pi}{(p^2 + l^2 + d^2)^{3/2}}\, dl = \frac{d/2\pi}{p^2 + d^2} \int_{-\infty}^{+\infty} \frac{d\left(\frac{l}{\sqrt{p^2+d^2}}\right)}{\left[1 + \left(\frac{l}{\sqrt{p^2+d^2}}\right)^2\right]^{3/2}}$$

$$= \frac{d/2\pi}{p^2 + d^2}\left[\frac{x}{(1 + x^2)^{1/2}}\right]_{-\infty}^{+\infty} = \frac{d/2\pi}{p^2 + d^2}.2$$

If the equation of the line is given by $ax+by+c = 0$, then the distance p of a point (x, y) to this line is given by a well-known formula as $p = (ax+by+c)/\sqrt{(}a^2+b^2)$. Herewith the light strength of a fuzzy image of a line is given by:

$$L(x, y) = \frac{d/\pi}{(ax + by + c)^2/(a^2 + b^2) + d^2}$$

This function is known from statistics as a *Cauchy distribution*. Again, for $d \to 0$, a crisp picture is obtained and the integral strength of the light is still equal to unity.
Quite in general, a crisp figure can be transformed into a fuzzy picture in the

following manner. Suppose that we know the fuzzy image of a single point. This "unit" image is called: $h(x)$. But, any crisp figure is imaged point by point. Let the crisp figure be called $f(x)$, then the light strength of only the function value $f(\xi)$ at the spot x will be $f(\xi)h(x-\xi)$. And this must be integrated over the whole image plane, in order to obtain the fuzzy image $\overline{f}$ of f:

$$\overline{f}(x) = \int_{-\infty}^{+\infty} f(\xi)h(x-\xi)\,d\xi = \int_{-\infty}^{+\infty} h(\xi)f(x-\xi)\,d\xi$$

This expression is known as a *convolution integral*. The derivation has has been given for the one-dimensional case. But the idea is also valid, of course, for the two dimensions of an image plane.
Suppose that, instead of fuzzyfying a line, we create a fuzzy image V of a half plane. The crisp picture in this case is a Heaviside or step function $H(p)$. Then with help of the above we find:

$$\int_{-\infty}^{+\infty} \frac{d/\pi}{\xi^2+d^2} H(p-\xi)\,d\xi = \frac{1}{\pi}\int_{-\infty}^{p} \frac{d\left(\frac{\xi}{d}\right)}{\left[1+\left(\frac{\xi}{d}\right)^2\right]}\,d\xi = \frac{1}{\pi} arctan(\frac{p}{d}) + \frac{1}{2}$$

Again, for $d \to 0$, a crisp picture $H(p)$ is obtained. And:

$$V(x,y) = \frac{1}{\pi} arctan(\frac{ax+by+c}{d\sqrt{a^2+b^2}}) + \frac{1}{2}$$

It's not suggested by the header of this section, but it's clear now that the making of fuzzy images is rather described by *exact* mathematics, using tools of classical analysis. Therefore, strictly speaking, there is nothing fuzzy about fuzzy optics!

Disclaimers

Anything free comes without referee :-(
My English may be better than your Dutch.